EPFL - Fall 2025	Dr. Pablo Antolin
Analysis III (for IC) – MATH-203(d) – IN SC	**Exercises**
Correction 8	November 6 2025

Exercise 1.
(i) Calculation of $\iiint_\Omega \operatorname{div} F dxdydz$. We have $\operatorname{div} F = 2(x+y+z)$. We set

$$x = ar\cos\theta,\ y = ar\sin\theta,\ z = bt, \quad \text{with } 0<\theta<2\pi,\ 0<r<t<1$$

and we therefore find

$$\text{Jacobian } = \begin{vmatrix} a\cos\theta & -ar\sin\theta & 0 \\ a\sin\theta & ar\cos\theta & 0 \\ 0 & 0 & b \end{vmatrix} = a^2br$$

This allows us to compute

$$\begin{aligned}\iiint_\Omega \operatorname{div} F dxdydz &= 2\int_0^{2\pi} d\theta \int_0^1 dt \int_0^t a^2br(ar\cos\theta + ar\sin\theta + bt)dr \\ &= 4\pi a^2b^2 \int_0^1 dt \int_0^t rtdr = \frac{\pi a^2 b^2}{2}\end{aligned}$$

(ii) Calculation of $\iint_{\partial\Omega} (F\cdot\nu)ds$. We have $\partial\Omega = \Sigma_1 \cup \Sigma_2$ where

$$\begin{aligned}\Sigma_1 &= \left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2 \le a^2,\ z=b\right\} \\ &= \{\alpha(r,\theta) = (r\cos\theta, r\sin\theta, b) : 0\le\theta\le 2\pi,\ 0\le r\le a\} \\ \Sigma_2 &= \left\{(x,y,z)\in\mathbb{R}^3 : b^2\left(x^2+y^2\right) = a^2z^2 \text{ and } 0\le z\le b\right\} \\ &= \{\beta(\theta,t) = (at\cos\theta, at\sin\theta, bt) : 0\le\theta\le 2\pi,\ 0\le t\le 1\}\end{aligned}$$

We find

$$\alpha_r \wedge \alpha_\theta = \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$

$$\beta_\theta \wedge \beta_t = \begin{vmatrix} e_1 & e_2 & e_3 \\ -at\sin\theta & at\cos\theta & 0 \\ a\cos\theta & a\sin\theta & b \end{vmatrix} = \begin{pmatrix} abt\cos\theta \\ abt\sin\theta \\ -a^2t \end{pmatrix}$$

which are both outward normals. We therefore obtain

$$\iint_{\Sigma_1} (F \cdot \nu) ds = \int_0^{2\pi} \int_0^a \left(r^2 \cos^2\theta, r^2 \sin^2\theta, b^2\right) \cdot (0,0,r) dr d\theta = \pi a^2 b^2$$

$$\begin{aligned}
&\iint_{\Sigma_2} (F \cdot \nu) ds \\
&= \int_0^{2\pi} \int_0^1 \left(a^2 t^2 \cos^2\theta, a^2 t^2 \sin^2\theta, b^2 t^2\right) \cdot \left(abt\cos\theta, abt\sin\theta, -a^2 t\right) dt d\theta \\
&= -2\pi a^2 b^2 \int_0^1 t^3 dt = -\frac{\pi}{2} a^2 b^2
\end{aligned}$$

and finally

$$\iint_{\partial\Omega} (F \cdot \nu) ds = \frac{\pi}{2} a^2 b^2$$

Exercise 2.
(i) Calculation of $\iiint_\Omega \operatorname{div} F dx dy dz$. We immediately have

$$\operatorname{div} F = x + y + z.$$

We have

$$\Omega = \left\{(x,y,z) \in \mathbb{R}^3 : 0 < z < 1 - x - y,\ 0 < y < 1 - x,\ 0 < x < 1\right\}$$

and therefore

$$\begin{aligned}
\iiint_\Omega \operatorname{div} F dx dy dz &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z) dz dy dx \\
&= \frac{1}{2} \int_0^1 \int_0^{1-x} \left(1 - (x+y)^2\right) dy dx = \frac{1}{8}
\end{aligned}$$

(ii) Calculation of $\iint_{\partial\Omega} (F \cdot \nu) ds$. We have $\partial\Omega = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$, where

$$\begin{aligned}
\Sigma_1 &= \{\alpha(x,z) = (x,0,z) : 0 \le z \le 1 - x,\ 0 \le x \le 1\} \\
\Sigma_2 &= \{\beta(x,y) = (x,y,0) : 0 \le y \le 1 - x,\ 0 \le x \le 1\} \\
\Sigma_3 &= \{\gamma(y,z) = (0,y,z) : 0 \le z \le 1 - y,\ 0 \le y \le 1\} \\
\Sigma_4 &= \{\delta(x,y) = (x,y,1-x-y) : 0 \le y \le 1 - x,\ 0 \le x \le 1\}
\end{aligned}$$

The corresponding normals are

$$\alpha_x \wedge \alpha_z = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\beta_x \wedge \beta_y = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\gamma_y \wedge \gamma_z = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\delta_x \wedge \delta_y = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

which are outward normals for the first and last, and inward normals for the other two. Note then that

$$\iint_{\Sigma_1} (F \cdot \nu) ds = \iint_{\Sigma_2} (F \cdot \nu) ds = \iint_{\Sigma_3} (F \cdot \nu) ds = 0$$

and therefore

$$\begin{aligned} \iint_{\partial\Omega} (F \cdot \nu) ds &= \iint_{\Sigma_4} (F \cdot \nu) ds \\ &= \int_0^1 \int_0^{1-x} ((xy, y(1-x-y), x(1-x-y)) \cdot (1,1,1) dy dx \end{aligned}$$

which finally gives us

$$\iint_{\partial\Omega} (F \cdot \nu) ds = \frac{1}{8}$$

Exercise 3.
(i) Calculation of $\iiint_\Omega \operatorname{div} F dxdydz$. We immediately find that $\operatorname{div} F = 3$. We switch to cylindrical coordinates and obtain

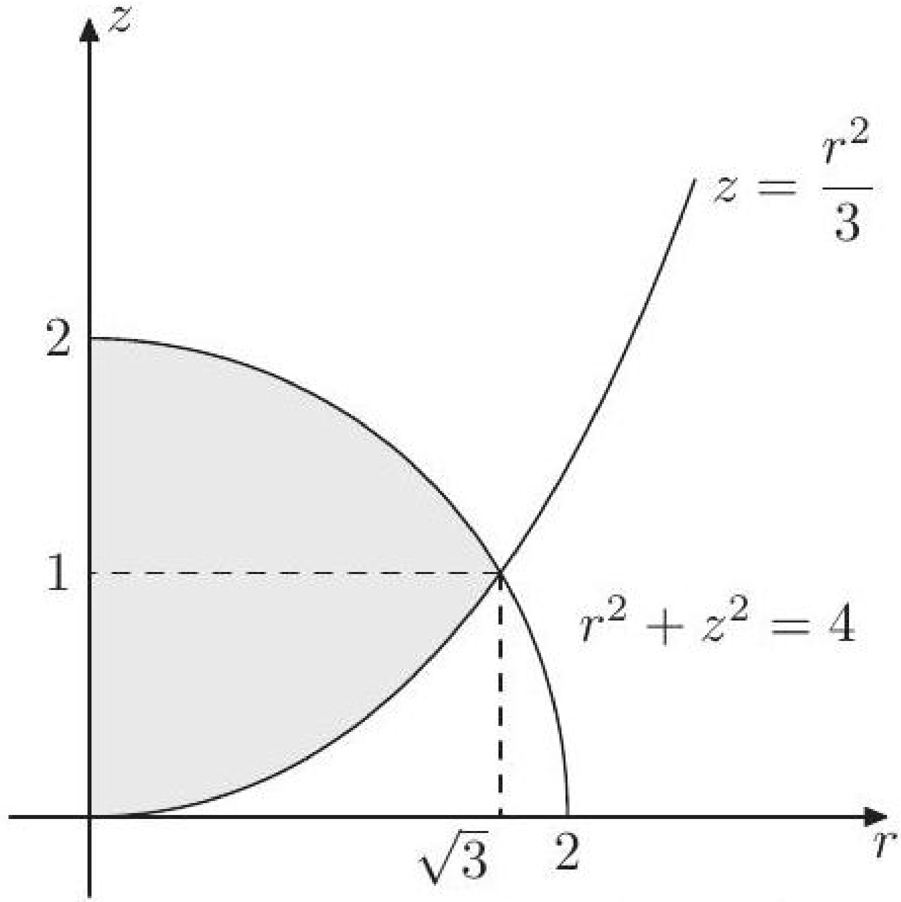
$$\Omega = \left\{ (r\cos\theta, r\sin\theta, z) : \theta \in (0, 2\pi),\ r \in (0, \sqrt{3}),\ \frac{r^2}{3} < z < \sqrt{4 - r^2} \right\}$$

and therefore

$$\iiint_\Omega \operatorname{div} F dxdydz = 3\int_0^{\sqrt{3}}\int_{\frac{r^2}{3}}^{\sqrt{4-r^2}}\int_0^{2\pi} r d\theta dz dr$$
$$= 6\pi\int_0^{\sqrt{3}}\left(\sqrt{4-r^2}-\frac{r^2}{3}\right) r dr = \frac{19\pi}{2}$$

(ii) Calculation of $\iint_{\partial\Omega}(F\cdot\nu)ds$. Note that $\partial\Omega = \Sigma_1\cup\Sigma_2$ where

$$\Sigma_1 = \left\{\alpha(\theta,\varphi) = (2\cos\theta\sin\varphi, 2\sin\theta\sin\varphi, 2\cos\varphi) : \theta\in[0,2\pi],\ \varphi\in\left[0,\frac{\pi}{3}\right]\right\}$$



(dessin dans le plan r, z)

$$\Sigma_2 = \left\{ \beta(r,\theta) = \left(r\cos\theta, r\sin\theta, \frac{r^2}{3} \right) : \theta \in [0, 2\pi],\ r \in [0, \sqrt{3}] \right\}$$

The calculation of the normals gives

$$\alpha_\theta \wedge \alpha_\varphi = \begin{vmatrix} e_1 & e_2 & e_3 \\ -2\sin\theta\sin\varphi & 2\cos\theta\sin\varphi & 0 \\ 2\cos\theta\cos\varphi & 2\sin\theta\cos\varphi & -2\sin\varphi \end{vmatrix} = \begin{pmatrix} -4\cos\theta\sin^2\varphi \\ -4\sin\theta\sin^2\varphi \\ -4\cos\varphi\sin\varphi \end{pmatrix}$$

$$\beta_r \wedge \beta_\theta = \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos\theta & \sin\theta & \frac{2}{3}r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \begin{pmatrix} -\frac{2}{3}r^2\cos\theta \\ -\frac{2}{3}r^2\sin\theta \\ r \end{pmatrix}$$

which are both inward normals. We can therefore compute the surface integrals

$$\begin{aligned}
&\iint_{\Sigma_1} (F\cdot\nu) ds = \int_0^{2\pi}\int_0^{\pi/3} [(2\cos\theta\sin\varphi, 2\sin\theta\sin\varphi, 2\cos\varphi) \\
&\left(4\cos\theta\sin^2\varphi, 4\sin\theta\sin^2\varphi, 4\cos\varphi\sin\varphi\right)]\, d\theta d\varphi \\
&= 16\pi \int_0^{\pi/3} \sin\varphi d\varphi = 16\pi[-\cos\varphi]_0^{\pi/3} = 8\pi \\
&\iint_{\Sigma_2} (F\cdot\nu) ds \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} \left(r\cos\theta, r\sin\theta, \frac{r^2}{3} \right) \cdot \left(\frac{2r^2\cos\theta}{3}, \frac{2r^2\sin\theta}{3}, -r \right) drd\theta \\
&= 2\pi \int_0^{\sqrt{3}} \left(\frac{2r^3}{3} - \frac{r^3}{3} \right) dr = 2\pi \left[\frac{r^4}{12} \right]_0^{\sqrt{3}} = \frac{3\pi}{2}
\end{aligned}$$

We therefore find

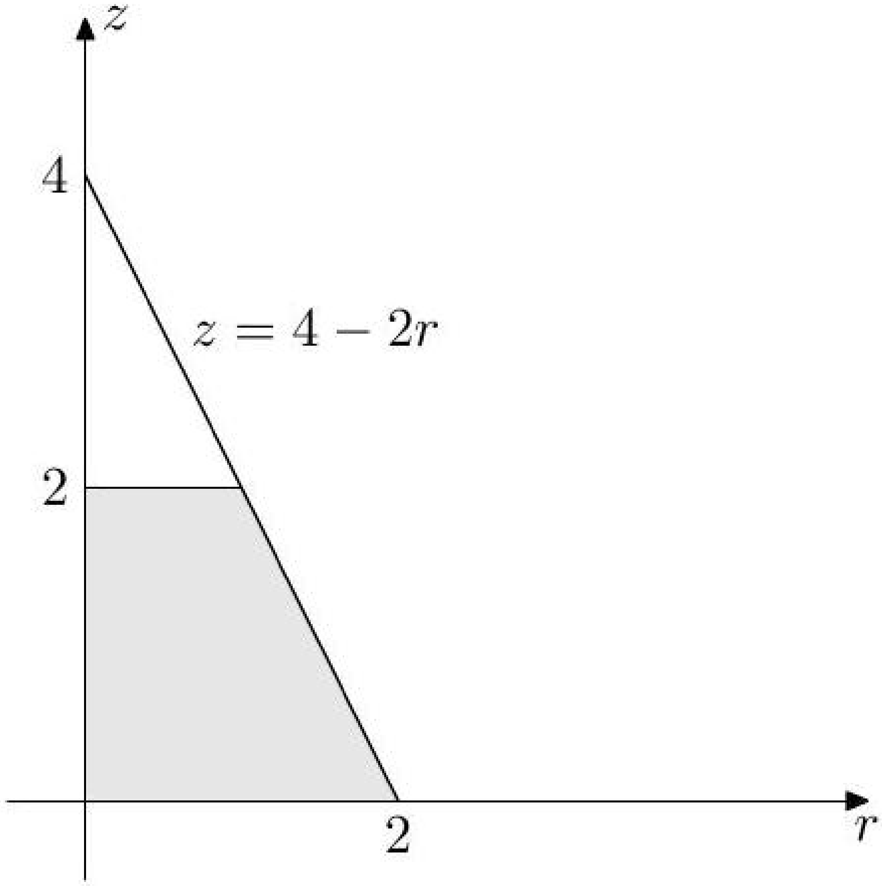
$$\iint_{\partial\Omega} (F\cdot\nu) ds = 8\pi + \frac{3\pi}{2} = \frac{19\pi}{2}$$

Exercise 4.
(i) Calculation of $\iiint_\Omega \operatorname{div} F dxdydz$. We find that $\operatorname{div} F = 2z$. Switching to cylindrical coordinates, we deduce that

$$\Omega = \left\{ (r\cos\theta, r\sin\theta, z) : 0 < z < 2,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ 0 < 2r < 4 - z \right\}$$

and therefore

$$\iiint_\Omega \operatorname{div} F dxdydz = \int_0^2 \int_0^{2-z/2} \int_{-\pi/2}^{\pi/2} 2zrd\theta dzdr$$
$$= \pi \int_0^2 z\left(2-\frac{z}{2}\right)^2 dz = \frac{11\pi}{3}$$



(drawing in the r, z plane with rotation angle $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$)
(ii) Calculation of $\iint_{\partial\Omega}(F\cdot\nu)ds$. We see that $\partial\Omega = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$ with

$$\Sigma_1 = \{\alpha(y,z) = (0,y,z) : z \in (0,2) \text{ and } 2|y| \le 4-z\}$$
$$\Sigma_2 = \left\{\beta(r,\theta) = (r\cos\theta, r\sin\theta, 0) : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ r \le 2\right\}$$
$$\Sigma_3 = \left\{\gamma(r,\theta) = (r\cos\theta, r\sin\theta, 2) : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ r \le 1\right\}$$
$$\Sigma_4 = \left\{\delta(r,\theta) = (r\cos\theta, r\sin\theta, 4-2r) : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 1 \le r \le 2\right\}.$$

The normals are then

$$\alpha_y \wedge \alpha_z = (1,0,0), \quad \beta_r \wedge \beta_\theta = (0,0,r)$$
$$\gamma_r \wedge \gamma_\theta = (0,0,r), \quad \delta_r \wedge \delta_\theta = (2r\cos\theta, 2r\sin\theta, r)$$

and these are inward normals for the first two and outward normals for the last two. Consequently, we find

$$\iint_{\Sigma_1} (F \cdot \nu) ds = \int_0^2 \int_{-2+z/2}^{2-z/2} \left(2, 0, z^2\right) \cdot (-1,0,0) dy dz = -12$$

$$\iint_{\Sigma_2} (F \cdot \nu) ds = \int_{-\pi/2}^{\pi/2} \int_0^2 \left(2, 0, r^3 \cos\theta \sin^2\theta\right) \cdot (0,0,-r) dr d\theta = -\frac{64}{15}$$

$$\iint_{\Sigma_3} (F \cdot \nu) ds = \int_{-\pi/2}^{\pi/2} \int_0^1 \left(2, 0, r^3 \cos\theta \sin^2\theta + 4\right) \cdot (0,0,r) dr d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \int_0^1 \left(r^4 \cos\theta \sin^2\theta + 4r\right) dr d\theta = \frac{2}{15} + 2\pi$$

$$\iint_{\Sigma_4} (F \cdot \nu) ds$$
$$= \int_{-\pi/2}^{\pi/2} \int_1^2 \left(2, 0, r^3 \cos\theta \sin^2\theta + (4-2r)^2\right) \cdot (2r\cos\theta, 2r\sin\theta, r) dr d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \int_1^2 \left(4r\cos\theta + r^4 \cos\theta \sin^2\theta + r(4-2r)^2\right) dr d\theta$$
$$= 12 + \frac{62}{15} + \frac{5\pi}{3}.$$

We thus obtain the desired result

$$\iint_{\partial\Omega} (F \cdot \nu) ds = \sum_{i=1}^{4} \iint_{\Sigma_i} (F \cdot \nu) ds = \frac{11\pi}{3}$$

Exercise 5.
We first note that

$$\operatorname{div} F = 3 \quad \text{and} \quad \operatorname{div} G_i = 1, \ i = 1, 2, 3.$$

Consequently, by applying the divergence theorem, we obtain, for $i = 1, 2, 3$,

$$\begin{aligned}\mathrm{Vol}(\Omega) &= \iiint_\Omega dxdydz = \frac{1}{3}\iiint_\Omega \operatorname{div} F dxdydz = \frac{1}{3}\iint_{\partial\Omega}(F\cdot\nu)ds \\ &= \iiint_\Omega \operatorname{div} G_i dxdydz = \iint_{\partial\Omega}(G_i\cdot\nu)\,ds\end{aligned}$$