EPFL - Fall 2025	Dr. Pablo Antolin
Analysis III (for IC) – MATH-203(d) – IN SC	**Exercises**
Correction 7	October 30 2025

Exercise 1.
Reminder: the *flux* of a regular orientable vector field F through the regular surface $\Sigma \subset \mathbb{R}^3$ is defined by:

$$\iint_\Sigma F \cdot \mathrm{d}S.$$

The sign of this expression is ambiguous if the direction of the continuous normal field (which exists since Σ is orientable) along Σ used is not specified.

Here, we have specified the choice of a unit normal field ν along Σ.

Let $\sigma : \bar{A} \mapsto \Sigma$ be a parameterization of Σ defined by $\sigma(u, v)$, **which respects the choice of orientation of** Σ, so that

$$\frac{\sigma_u \wedge \sigma_v}{\|\sigma_u \wedge \sigma_v\|} = \nu(u, v).$$

Using the definitions from the course for surface integrals of a scalar field and a vector field, we have:

$$\begin{aligned}\iint_\Sigma (F \cdot \nu)\, \mathrm{d}S &= \iint_A [F(\sigma(u,v)) \cdot \nu(u,v)]\, \|\sigma_u \wedge \sigma_v\|\, \mathrm{d}u\mathrm{d}v \\ &= \iint_A [F(\sigma(u,v)) \cdot \sigma_u \wedge \sigma_v]\, \mathrm{d}u\mathrm{d}v \\ &= \iint_\Sigma F \cdot \mathrm{d}S.\end{aligned}$$

It should be noted that the sign of a surface integral of a scalar field is not ambiguous (we take the norm of $\sigma_u \wedge \sigma_v$, its direction does not matter). Here, and as in the divergence theorem, it is the explicit choice of a normal unit ν that fixes the sign.

Exercise 2.

$$\sigma(\theta, z) = (z\cos\theta, z\sin\theta, z), \quad \text{with } (\theta, z) \in A = (0, 2\pi) \times (0, 1).$$

The normal is then given by

$$\sigma_\theta \wedge \sigma_z = (z\cos\theta, z\sin\theta, -z) \quad \Rightarrow \quad \|\sigma_\theta \wedge \sigma_z\| = \sqrt{2}z.$$

We therefore find

$$\iint_\Sigma f\, ds = \int_0^1 \int_0^{2\pi} \sqrt{2}z \left(z^2 \cos\theta \sin\theta + z^2\right) d\theta dz = 2\pi\sqrt{2} \int_0^1 z^3 dz = \frac{\pi}{\sqrt{2}}$$

Exercise 3.
We take again the parametrization from Series 6: $\sigma \colon [0,1] \times [0, 2\pi] \to \mathbb{R}^3$ defined by

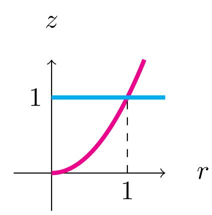
$$\begin{aligned} \sigma(r, \theta) &= (r\cos\theta, r\sin\theta, r) \\ \|\sigma_r \wedge \sigma_\theta\| &= \sqrt{2}r \end{aligned}$$

thus,

$$\begin{aligned} \rho(\sigma(r,\theta)) &= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r \\ \int_\Sigma \rho\, ds &= \int_0^{2\pi} \int_0^1 \sqrt{2}r^2 dr d\theta \\ &= \frac{2\sqrt{2}\pi}{3} \end{aligned}$$

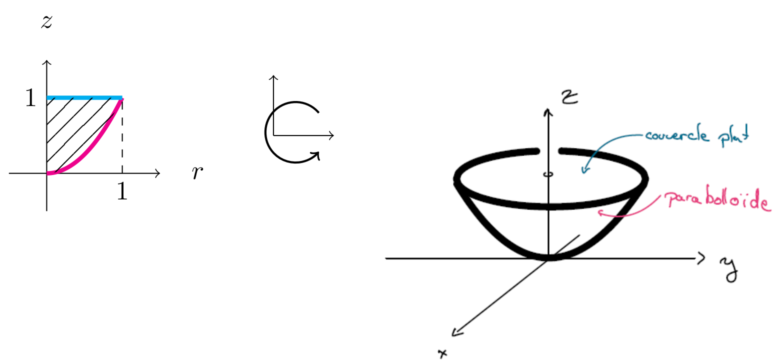
Exercise 4.
We identify the cylindrical symmetry and move to the corresponding coordinates: $(x, y, z) = (r\cos\theta, r\sin\theta, z)$ with $r \geq 0$, $\theta \in [0, 2\pi]$, and $z \in \mathbb{R}$. Our conditions become

$$\begin{aligned} x^2 + y^2 \leq z \quad &\Leftrightarrow \quad r^2 \leq z \\ z \leq 1 \quad &\Leftrightarrow \quad \text{unchanged} \end{aligned}$$



where we found the coordinates of the intersection point by solving the system

$$\begin{cases} r^2 = z \\ z = 1 \end{cases}$$



The boundary of Ω therefore splits into two parts: Σ_1, the parabolic part where $r^2 = z$, and Σ_2, the flat lid where $z = 1$.

$$\begin{aligned}
&\Sigma_1 : \\
&\qquad \sigma\colon [0,1] \times [0, 2\pi] \to \mathbb{R}^3 \\
\sigma(r,\theta) =& (r\cos\theta, r\sin\theta, r^2) \\
\sigma_r \wedge \sigma_\theta =& \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix} \\
\|\sigma_r \wedge \sigma_\theta\| =& \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1} \\
\text{Area}(\Sigma_1) =& \iint_{\Sigma_1} 1\, ds = \int_0^{2\pi} \int_0^1 r\sqrt{4r^2+1}\, dr\, d\theta \\
=& 2\pi \left[\frac{1}{12} (1+4r^2)^{\frac{3}{2}} \right]_0^1 \\
=& \frac{\pi}{6} \left(5^{\frac{3}{2}} - 1 \right) = \frac{(5\sqrt{5}-1)\pi}{6}
\end{aligned}$$

$$\Sigma_2:$$

$$\begin{aligned}
\sigma\colon [0,1]\times[0,2\pi] &\to \mathbb{R}^3\\
\sigma(r,\theta) =&(r\cos\theta, r\sin\theta, 1)\\
\sigma_r\wedge\sigma_\theta = &\begin{vmatrix} e_1 & e_2 & e_3\\ \cos\theta & \sin\theta & 0\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix} = \begin{pmatrix}0\\0\\r\end{pmatrix}\\
\|\sigma_r\wedge\sigma_\theta\| &= r\\
\text{Area}(\Sigma_2) &= \iint_{\Sigma_2} 1\,ds = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \pi.
\end{aligned}$$

(We could also have noticed that Σ_2 is a disk of radius 1 and used the formula πR^2.)

Finally,

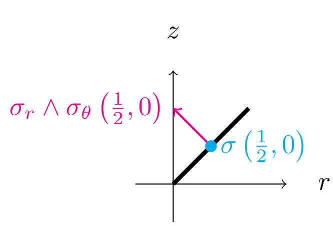
$$\text{Area}(\partial\Omega) = \text{Area}(\Sigma_1) + \text{Area}(\Sigma_2) = \frac{(5\sqrt{5}-1)\pi}{6} + \pi = \frac{5(\sqrt{5}+1)\pi}{6}$$

Exercise 5.
We reuse the parametrization from Exercise 3 of this series or from series 6: $\sigma\colon [0,1]\times[0,2\pi]\to\mathbb{R}^3$ defined by

$$\begin{aligned}
\sigma(r,\theta) =&(r\cos\theta, r\sin\theta, r)\\
\sigma_r\wedge\sigma_\theta = &\begin{pmatrix}-r\cos\theta\\ -r\sin\theta\\ r\end{pmatrix}
\end{aligned}$$

To determine whether the normal vector $\sigma_r\wedge\sigma_\theta$ is upward or downward, we can either observe that the third component is always positive, and therefore it points upward and is ascending, or verify it graphically. We examine what happens at $r=\frac{1}{2}$, $\theta=0$:

in the plane $\theta = 0$, i.e. in the plane $y=0$, $x\geq 0$



Since the vector points upward, the direction is ascending.

Finally,

$$
\begin{aligned}
F(\sigma(r,\theta)) &= \left(r^2\cos^2\theta, r^2\sin^2\theta, r^2\right)\\
\iint_\Sigma F\cdot ds &= \int_0^{2\pi}\int_0^1 \langle F(\sigma(r,\theta)), \sigma_r\wedge\sigma_\theta\rangle\, drd\theta\\
&= \int_0^{2\pi}\int_0^1 \left\langle \left(r^2\cos^2\theta, r^2\sin^2\theta, r^2\right), \left(-r\cos\theta, -r\sin\theta, r\right)\right\rangle drd\theta\\
&= \int_0^{2\pi}\int_0^1 -r^3\cos^3\theta - r^3\sin^3\theta + r^3 drd\theta\\
&= \int_0^{2\pi} -\frac{1}{4}\cos^3\theta - \frac{1}{4}\sin^3\theta + \frac{1}{4}d\theta\\
&= -\frac{1}{4}\int_0^{2\pi}(1-\sin^2\theta)\cos\theta + (1-\cos^2\theta)\sin\theta d\theta + \frac{\pi}{2}\\
&= -\frac{1}{4}\int_0^{2\pi}\cos\theta - \sin^2\theta\cos\theta + \sin\theta - \cos^2\theta\sin\theta d\theta\\
&=\frac{1}{4}\left[\sin\theta + \frac{1}{3}\sin^3\theta - \cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{2\pi} + \frac{\pi}{2}\\
&=\frac{\pi}{2}.
\end{aligned}
$$

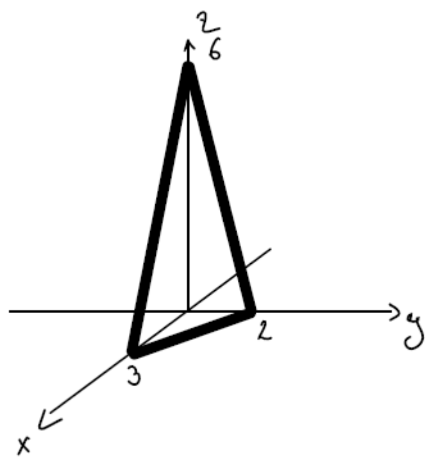
where we used that $\cos^2\theta = 1-\sin^2\theta$ and $\sin^2\theta = 1-\cos^2\theta$. Alternatively, we can compute the antiderivative of $\cos^3\theta$ and $\sin^3\theta$ using Euler's formulas:

$$
\begin{aligned}
\cos^3\theta &= \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^3 = \frac{1}{8}\left(e^{3i\theta}+3e^{i\theta}+3e^{-i\theta}+e^{-3i\theta}\right)\\
&=\frac{1}{4}\frac{e^{3i\theta}+e^{-3i\theta}}{2} + \frac{3}{4}\frac{e^{i\theta}+e^{-i\theta}}{2}\\
&=\frac{1}{4}\cos(3\theta) + \frac{3}{4}\cos(\theta)\\
\int\cos^3\theta d\theta &=\frac{1}{12}\sin(3\theta) + \frac{3}{4}\sin(\theta)
\end{aligned}
$$

$$
\begin{aligned}
\sin^3\theta &= \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3 = -\frac{1}{8i}\left(e^{3i\theta}-3e^{i\theta}+3e^{-i\theta}-e^{-3i\theta}\right)\\
&= -\frac{1}{4}\frac{e^{3i\theta}-e^{-3i\theta}}{2i} + \frac{3}{4}\frac{e^{i\theta}-e^{-i\theta}}{2i}\\
&= -\frac{1}{4}\sin(3\theta) + \frac{3}{4}\sin(\theta)\\
\int\sin^3\theta d\theta &=\frac{1}{2}\cos(3\theta) - \frac{3}{4}\cos(\theta)
\end{aligned}
$$

Exercise 6.
Since the equation $z = 6 - 3x - 2y$ is affine, we expect the surface to be part of a plane. Let's see where the plane intersects the axes in order to reconstruct the surface:

$$\begin{aligned} (0,0,z) \in \Sigma \quad &\Leftrightarrow \quad z = 6 \\ (0,y,0) \in \Sigma \quad &\Leftrightarrow \quad y = 3 \\ (x,0,0) \in \Sigma \quad &\Leftrightarrow \quad x = 2 \end{aligned}$$



To find the domain of the parametrization, we proceed as follows:

$$6 - 3x - 2y = z \geq 0 \quad \Rightarrow \quad 3x + 2y \leq 6 \quad \Rightarrow \quad y \leq 3 - \frac{3}{2}x.$$

By transitivity, we also obtain

$$0 \leq y \leq 3 - \frac{3}{2}x \quad \Rightarrow \quad \frac{3}{2}x \leq 3 \quad \Rightarrow \quad x \in [0,2].$$

Finally, a parametrization of Σ is $\sigma\colon \left\{(x,y) \in \mathbb{R}^2 \ : \ x \in [0,2],\ 0 \leq y \leq 3 - \frac{3}{2}x\right\}$ defined by

$$\sigma(x,y) = (x, y, 6 - 3x - 2y).$$

The normal induced by this parametrization is

$$\sigma_x \wedge \sigma_y = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{vmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Since the third component is positive, the normal is upward and therefore points in the correct direction. Finally, we have

$$\begin{aligned}
F(\sigma(x,y)) =& (0, 6-3x-2y, 6-3x-2y) \\
\int_\Sigma F \cdot ds =& \int_0^2 \int_0^{3-\frac{3}{2}x} \langle (0, 6-3x-2y, 6-3x-2y), (3,2,1) \rangle \, dydx \\
=& \int_0^2 \int_0^{3-\frac{3}{2}x} 18 - 9x - 6y dydx \\
=& \int_0^2 \left[18y - 9xy - 3y^2\right]_{y=0}^{y=3-\frac{3}{2}x} dx \\
=& \int_0^2 54 - 27x - 27x + \frac{27}{2}x^2 - 3\left(3 - \frac{3}{2}x\right)^2 dx \\
=& \int_0^2 54 - 54x + \frac{27}{2}x^2 - 3\left(9 - 9x + \frac{9}{4}x^2\right) dx \\
=& \int_0^2 54 - 54x + \frac{27}{2}x^2 - 27 + 27x - \frac{27}{4}x^2 dx \\
=& \int_0^2 27 - 27x + \frac{27}{4}x^2 dx \\
=& \left[27x - \frac{27}{2}x^2 + \frac{9}{4}x^3\right]_0^2 \\
=& 54 - 54 + 18 = 18
\end{aligned}$$

Exercise 7.
Variant 1: f is inside the domain of the parametrization.

A parametrization of Σ can be $\sigma \colon \left\{(t,z) \in \mathbb{R}^2 \; : \; t \in [a,b], \; 0 \leq z \leq f(\gamma(t))\right\}$ defined by

$$\sigma(t,z) = (\gamma_1(t), \gamma_2(t), z), \qquad t \in [a,b], z \in [0, f(\gamma(t))].$$

We then have

$$\begin{aligned}
\sigma_t =& (\gamma_1'(t), \gamma_2'(t), 0) \\
\sigma_z =& (0,0,1) \\
\sigma_t \wedge \sigma_z =& \begin{vmatrix} e_1 & e_2 & e_3 \\ \gamma_1'(t) & \gamma_2'(t) & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\gamma_2'(t), -\gamma_1'(t), 0) \\
\|\sigma_t \wedge \sigma_z\| =& \|\gamma'(t)\|.
\end{aligned}$$

Thus,

$$\text{Area}(\Sigma) = \int_\Sigma 1 ds = \int_a^b \int_0^{f(\gamma(t))} \|\sigma_t \wedge \sigma_z\| dzdt = \int_a^b f(\gamma(t)) \|\gamma'(t)\| dt = \int_\Gamma f dl.$$

Variant 2: f appears in the expression of the parametrization.

A parametrization of Σ can be $\sigma\colon [a,b]\times[0,1]\to\mathbb{R}^3$ defined by

$$\sigma(t,h) = (\gamma_1(t), \gamma_2(t), h\cdot f(\gamma(t)))$$

We then have

$$\begin{aligned}
\sigma_t =& (\gamma_1'(t), \gamma_2'(t), h\langle \nabla f(\gamma(t)), \gamma'(t)\rangle)\\
\sigma_h =& (0,0,f(\gamma(t)))\\
\sigma_t\wedge\sigma_h =& \begin{vmatrix} e_1 & e_2 & e_3\\ \gamma_1'(t) & \gamma_2'(t) & \langle\nabla f(\gamma(t)),\gamma'(t)\rangle\\ 0 & 0 & f(\gamma(t))\end{vmatrix} = (\gamma_2'(t)f(\gamma(t)), -\gamma_1'(t)f(\gamma(t)), 0)\\
\|\sigma_t\wedge\sigma_z\| =& |f(\gamma(t))|\cdot\|\gamma'(t)\| \overset{f\geq 0}{=} f(\gamma(t))\,\|\gamma'(t)\|
\end{aligned}$$

Thus,

$$\text{Area}(\Sigma) = \int_\Sigma 1 ds = \int_a^b\int_0^1 \|\sigma_t\wedge\sigma_h\|\, dhdt = \int_a^b f(\gamma(t))\,\|\gamma'(t)\|\, dt = \int_\Gamma f dl$$

Exercise 8.
By the divergence theorem, the quantity

$$\iint_S F_{\alpha,\beta}\cdot \mathrm{d}S$$

is equal to

$$\iiint_{B_1} \operatorname{div} F_{\alpha,\beta}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

up to a sign. (where B_1 is the ball of radius 1). Moreover,

$$\operatorname{div} F_{\alpha,\beta}(x,y,z) = \frac{1}{(y^2+z^2)^\alpha} + 1 + \frac{1}{|x|^\beta}.$$

Thus, we only need to determine for which values of α and β we obtain

$$\iiint_{B_1}\frac{1}{(y^2+z^2)^\alpha}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z < +\infty \quad\text{and}\quad \iiint_{B_1}\frac{1}{|x|^\beta}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z < +\infty. \tag{1}$$

Now note that $B_1\subset\{(x,y,z)\in\mathbb{R}^3 : y^2+z^2<1\}$ and thus, since the function

$\frac{1}{(y^2+z^2)^\alpha}$ is always positive, we have

$$\begin{aligned}\iiint_{B_1} \frac{1}{(y^2+z^2)^\alpha}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &\le \iiint_{\{(x,y,z)\in\mathbb{R}^3 : y^2+z^2<1\}} \frac{1}{(y^2+z^2)^\alpha}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &\le 2\iint_{\{(y,z)\in\mathbb{R}^3 : y^2+z^2<1\}} \frac{1}{(y^2+z^2)^\alpha}\,\mathrm{d}y\mathrm{d}z \\ &= 2\int_0^{2\pi}\int_0^1 \frac{1}{r^{2\alpha}} r\,dr d\theta \\ &= 4\pi \int_0^1 r^{1-2\alpha}\,dr.\end{aligned}$$

This last integral is finite if and only if $\alpha < 1$. Similarly, for the second integral of (**??**), we have $B_1 \subset \{(x,y,z)\in\mathbb{R}^3 : |x|,|y|,|z|<1\}$. Since the function $\frac{1}{|x|^\beta}$ is positive, we get

$$\begin{aligned}\iiint_{B_1} \frac{1}{|x|^\beta}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &\le \iiint_{\{(x,y,z)\in\mathbb{R}^3 : |x|,|y|,|z|<1\}} \frac{1}{|x|^\beta}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &\le 4\int_{-1}^1 \frac{1}{|x|^\beta}\,\mathrm{d}x \\ &= 8\int_0^1 s^{-\beta}\,ds\end{aligned}$$

Here the last integral is finite if and only if $\beta < 1$. We conclude that

$$\left|\iint_S F_{\alpha,\beta}\,\mathrm{d}S\right| < +\infty$$

if $\alpha < 1$ and $\beta < 1$.

Remark.
We have not shown that

$$\left|\iint_S F_{\alpha,\beta}\,\mathrm{d}S\right| < +\infty$$

if and only if $\alpha < 1$ and $\beta < 1$. We have only shown that if $\alpha < 1$ and $\beta < 1$, then the above integral is finite. However, it is also true that if

$$\left|\iint_S F_{\alpha,\beta}\,\mathrm{d}S\right| < +\infty,$$

then $\alpha < 1$ and $\beta < 1$. To show this, it is sufficient to prove that if $\alpha \ge 1$ or $\beta \ge 1$, then

$$\left|\iint_S F_{\alpha,\beta}\,\mathrm{d}S\right| = +\infty. \tag{2}$$

Suppose $\alpha \ge 1$ (the case $\beta \ge 1$ is similar). By taking a cylinder (with small radius and small height) aligned with the Ox axis such that it is contained within

B_1, we find (**??**) using the same arguments as above. This is nevertheless not the goal of the exercise.