

Exercise 1.

1. Let us note that

$$\operatorname{rot} \Phi = \frac{\partial}{\partial x} [\Phi_2] - \frac{\partial}{\partial y} [\Phi_1] = \frac{\partial}{\partial x} [F_1] - \frac{\partial}{\partial y} [-F_2] = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \operatorname{div} F.$$

Thus, by Green's theorem,

$$\iint_{\Omega} \operatorname{div} F(x, y) dx dy = \iint_{\Omega} \operatorname{rot} \Phi(x, y) dx dy = \int_{\partial\Omega} \Phi \cdot dl$$

Moreover, if $\gamma = (\gamma_1, \gamma_2): [a, b] \rightarrow \mathbb{R}^2$ is a parametrization that keeps the domain on the left (i.e. $\partial\Omega$ is positively oriented), we have

$$\begin{aligned} \int_{\partial\Omega} \Phi \cdot dl &= \int_a^b \langle \Phi(\gamma(t)), \gamma'(t) \rangle dt = \int_a^b \langle (-F_2(\gamma(t)), F_1(\gamma(t))), (\gamma_1'(t), \gamma_2'(t)) \rangle dt \\ &= \int_a^b F_1(\gamma(t))\gamma_2'(t) - F_2(\gamma(t))\gamma_1'(t) dt \end{aligned}$$

Furthermore, using the formula for computing a line integral of $\langle F, \nu \rangle$, we have

$$\begin{aligned} \int_{\partial\Omega} \langle F, \nu \rangle dl &= \int_a^b \langle (F_1(\gamma(t)), F_2(\gamma(t))), (\gamma_2'(t), -\gamma_1'(t)) \rangle dt \\ &= \int_a^b F_1(\gamma(t))\gamma_2'(t) - F_2(\gamma(t))\gamma_1'(t) dt = \int_{\partial\Omega} \Phi \cdot dl \end{aligned}$$

Hence, the result follows.

2. By applying the divergence theorem to $F = \nabla f$, we have

$$\iint_{\Omega} \Delta f(x, y) dx dy = \int_{\Omega} \operatorname{div} (\nabla f)(x, y) dx dy = \int_{\partial\Omega} \langle \nabla f, \nu \rangle dl,$$

which is the desired result.

Exercise 2.

We switch to cylindrical coordinates: $(x, y, z) = (r \cos \theta, r \sin \theta, z)$, with $r \geq 0$, $\theta \in [0, 2\pi]$, and $z \in \mathbb{R}$.

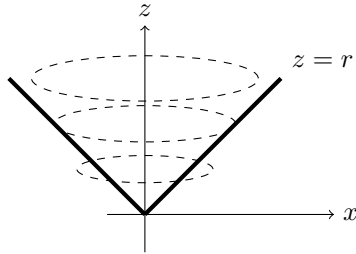
Our conditions become:

$$x^2 + y^2 = z^2 \Leftrightarrow r^2 = z^2 \Leftrightarrow r = |z| \stackrel{z \geq 0}{=} z$$

$$0 \leq z \leq 1 \text{ remains unchanged.}$$

and therefore a parametrization of the surface is given by $\sigma: [0, 1] \times [0, 2\pi] \rightarrow \mathbb{R}^3$ defined by

$$\sigma(r, \theta) = (r \cos \theta, r \sin \theta, r).$$



Moreover, we have

$$\begin{aligned} \sigma_r \wedge \sigma_\theta &= \begin{vmatrix} e_1 & e_2 & e_3 \\ \sigma_r^1 & \sigma_r^2 & \sigma_r^3 \\ \sigma_\theta^1 & \sigma_\theta^2 & \sigma_\theta^3 \end{vmatrix} \\ &= \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos \theta & \sin \theta & 1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} \\ &= e_1(-r \cos \theta) + e_2(-r \sin \theta) + e_3(r \cos^2 \theta + r \sin^2 \theta) \\ &= \begin{pmatrix} -r \cos \theta \\ -r \sin \theta \\ r \end{pmatrix} \end{aligned}$$

and therefore,

$$\|\sigma_r \wedge \sigma_\theta\| = \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta + r^2} = \sqrt{2}r.$$

Finally,

$$\begin{aligned}
 f(\sigma(r, \theta)) &= r^2 \cos \theta \sin \theta + r^2 \\
 \iint_{\Sigma} f \, ds &= \int_0^{2\pi} \int_0^1 (r^2 \cos \theta \sin \theta + r^2) \sqrt{2}r \, dr \, d\theta \\
 &= \sqrt{2} \int_0^{2\pi} \int_0^1 r^3 \cos \theta \sin \theta + r^3 \, dr \, d\theta \\
 &= \sqrt{2} \int_0^{2\pi} \frac{1}{4} \cos \theta \sin \theta + \frac{1}{4} \, d\theta \\
 &= \frac{\sqrt{2}}{4} \left[\frac{1}{2} \sin^2 \theta + \theta \right]_0^{2\pi} \\
 &= \frac{\sqrt{2}\pi}{2}
 \end{aligned}$$

Exercise 3.

1. For a 3D representation of Ω , see <https://www.geogebra.org/calculator/ctqgfts5>.
2. The parametrization is $u: [0, a] \times [0, 2\pi] \times [0, 2\pi] \rightarrow \mathbb{R}^3$ defined by

$$u(r, \theta, \varphi) = ((R + r \cos \varphi) \cos \theta, (R + r \cos \varphi) \sin \theta, r \sin \varphi).$$

Thus, the Jacobian matrix of φ is

$$\nabla u(r, \theta, \varphi) = \begin{pmatrix} \cos \varphi \cos \theta & -(R + r \cos \varphi) \sin \theta & -r \sin \varphi \cos \theta \\ \cos \varphi \sin \theta & (R + r \cos \varphi) \cos \theta & -r \sin \varphi \sin \theta \\ \sin \varphi & 0 & r \cos \varphi \end{pmatrix}$$

and therefore the Jacobian determinant of u is

$$\begin{aligned}
 \det \nabla u(r, \theta, \varphi) &= r(R + \cos \varphi) \cos^2 \varphi \cos^2 \theta + r(R + r \cos \varphi) \sin^2 \varphi \sin^2 \theta \\
 &\quad + r(R + r \cos \varphi) \sin^2 \varphi \cos^2 \theta + r(R + r \cos \varphi) \cos^2 \varphi \sin^2 \theta \\
 &= r(R + r \cos \varphi) (\cos^2 \varphi (\cos^2 \theta + \sin^2 \theta) + \sin^2 \varphi (\cos^2 \theta + \sin^2 \theta)) \\
 &= r(R + r \cos \varphi).
 \end{aligned}$$

Note that since $0 \leq r \leq a \leq R$ and $\cos \varphi \in [-1, 1]$, we have $R + r \cos \varphi \geq 0$, and thus

$$|\det \nabla u(r, \theta, \varphi)| = r(R + r \cos \varphi).$$

3. We have

$$\begin{aligned}\text{vol}(\Omega) &= \iiint_{\Omega} dx dy dz = \int_0^a \int_0^{2\pi} \int_0^{2\pi} r(R + r \cos \varphi) d\theta d\varphi dr \\ &= 2\pi \int_0^a [r(R\varphi + r \sin \varphi)]_0^{2\pi} = 4\pi^2 R \frac{a^2}{2} = 2\pi^2 R a^2.\end{aligned}$$

4. The parametrization of the surface of the torus consists of fixing $r = a$. Thus, $\sigma: [0, 2\pi] \times [0, 2\pi] \rightarrow \mathbb{R}^3$ is defined by

$$\sigma(\theta, \varphi) = u(a, \theta, \varphi) = ((R + a \cos \varphi) \cos \theta, (R + a \cos \varphi) \sin \theta, a \sin \varphi),$$

and we have

$$\begin{aligned}\sigma_{\theta}(\theta, \varphi) &= -(R + a \cos \varphi) \sin \theta, (R + a \cos \varphi) \cos \theta, 0), \\ \sigma_{\varphi}(\theta, \varphi) &= (-a \sin \varphi \cos \theta, -a \sin \varphi \sin \theta, a \cos \varphi),\end{aligned}$$

$$\begin{aligned}\sigma_{\theta} \wedge \sigma_{\varphi} &= \begin{vmatrix} e_1 & e_2 & e_3 \\ -(R + a \cos \varphi) \sin \theta & (R + a \cos \varphi) \cos \theta & 0 \\ -a \sin \varphi \cos \theta & -a \sin \varphi \sin \theta & a \cos \varphi \end{vmatrix} \\ &= \begin{pmatrix} a(R + a \cos \varphi) \cos \varphi \cos \theta \\ a(R + a \cos \varphi) \cos \varphi \sin \theta \\ a(R + a \cos \varphi) \sin \varphi \sin^2 \theta + a(R + a \cos \varphi) \sin \varphi \cos^2 \theta \end{pmatrix} \\ &= a(R + a \cos \varphi) \begin{pmatrix} \cos \varphi \cos \theta \\ \cos \varphi \sin \theta \\ \sin \varphi \end{pmatrix},\end{aligned}$$

which is a normal vector to the surface.

5. We have

$$\begin{aligned}|\sigma_{\theta} \wedge \sigma_{\varphi}| &= a(R + a \cos \varphi) \sqrt{\cos^2 \varphi \cos^2 \theta + \cos^2 \varphi \sin^2 \theta + \sin^2 \varphi} = a(R + a \cos \varphi), \\ \text{Area}(\partial\Omega) &= \iint_{\partial\Omega} ds = \int_0^{2\pi} \int_0^{2\pi} a(R + a \cos \varphi) d\theta d\varphi \\ &= 2\pi [a(R\varphi + a \sin \varphi)]_0^{2\pi} = 4\pi^2 aR.\end{aligned}$$

6. We have

$$\begin{aligned}\iiint_{\Omega} z^2 dx dy dz &= \int_0^a \int_0^{2\pi} \int_0^{2\pi} r^2 \sin^2 \varphi \cdot r(R + r \cos \varphi) d\theta d\varphi dr \\ &= 2\pi \int_0^a \int_0^{2\pi} Rr^3 \sin^2 \varphi + r^3 \sin^2 \varphi \cos \varphi d\varphi dr.\end{aligned}$$

Using that

$$\begin{aligned}\int \sin^2 \varphi d\varphi &= \frac{1}{2} (\varphi - \sin \varphi \cos \varphi), \\ \int \sin^2 \varphi \cos \varphi d\varphi &= \frac{1}{3} \sin^3 \varphi,\end{aligned}$$

or, using Euler's formulas:

$$\begin{aligned}\sin^2 \varphi &= \left(\frac{e^{i\varphi} - e^{-i\varphi}}{2i} \right)^2 = -\frac{e^{i2\varphi} - 2 + e^{-i2\varphi}}{4} = \frac{1}{2} - \frac{1}{2} \cos(2\varphi), \\ \int \sin^2 \varphi \, d\varphi &= \frac{\varphi}{2} - \frac{1}{4} \sin(2\varphi), \\ \sin^2 \varphi \cos \varphi \, d\varphi &= \left(\frac{e^{i\varphi} - e^{-i\varphi}}{2i} \right)^2 \frac{e^{i\varphi} + e^{-i\varphi}}{2} = -\frac{e^{i2\varphi} - 2 + e^{-i2\varphi}}{4} \frac{e^{i\varphi} + e^{-i\varphi}}{2} \\ &= -\frac{1}{8} (e^{i3\varphi} - 2e^{i\varphi} + e^{-i\varphi} + e^{i\varphi} - 2e^{-i\varphi} + e^{-i3\varphi}) \\ &= -\frac{1}{8} (e^{i3\varphi} + e^{-i3\varphi} - e^{i\varphi} - e^{-i\varphi}) \\ &= \frac{1}{4} \cos \varphi - \frac{1}{4} \cos(3\varphi), \\ \int \sin^2 \varphi \cos \varphi \, d\varphi &= \frac{1}{4} \sin \varphi - \frac{1}{12} \sin(3\varphi).\end{aligned}$$

Thus, returning to our computation,

$$\iiint_{\Omega} z^2 \, dx \, dy \, dz = 2\pi \int_0^a \left[Rr^3 \frac{1}{2} (\varphi - \sin \varphi \cos \varphi) + r^3 \frac{1}{3} \sin^3 \varphi \right]_0^{2\pi} = 2\pi^2 R \int_0^a r^3 \, dr = \frac{\pi^2 R a^4}{2}.$$

Exercise 4.

1. D_1 is the triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$. We can parameterize it as $D_1 = \{(x, y) \in \mathbb{R}^2 : x \in [0, 1], y \in [0, 1 - x]\}$.

$$\begin{aligned}\iint_{D_1} \sqrt{1-x-y} \, dx \, dy &= \int_0^1 dx \int_0^{1-x} \sqrt{1-x-y} \, dy \\ &= \int_0^1 \left(-\frac{2}{3} (1-x-y)^{3/2} \Big|_{y=0}^{y=1-x} \right) dx \\ &= \frac{2}{3} \int_0^1 (1-x)^{3/2} \, dx = \frac{2}{3} \cdot \left(-\frac{2}{5} (1-x)^{5/2} \Big|_{x=0}^{x=1} \right) \\ &= \frac{2}{3} \cdot \frac{2}{5} = \frac{4}{15}.\end{aligned}$$

2. To parameterize D_2 , we use polar coordinates, $x = r \cos \theta$ and $y = r \sin \theta$, with $\theta \in [0, 2\pi]$ and $r \in [0, 2(1 + \cos \theta)]$. Indeed, observe that

$$0 \leq x^2 + y^2 \leq 2 \left(x + \sqrt{x^2 + y^2} \right) \Leftrightarrow 0 \leq r \leq 2(1 + \cos \theta).$$

Recall that for polar coordinates, the Jacobian is r .

$$\iint_{D_2} \frac{dx dy}{(x^2 + y^2)^{\frac{3}{4}}} = \int_0^{2\pi} d\theta \int_0^{2(1+\cos\theta)} \frac{1}{r^{3/2}} r dr = \int_0^{2\pi} 2\sqrt{2(1+\cos\theta)} d\theta$$

using the trigonometric identity $1 + \cos\theta = 2\cos^2(\theta/2)$, we find

$$\begin{aligned} &= 4 \int_0^{2\pi} |\cos(\theta/2)| d\theta = 4 \int_0^{\pi} \cos(\theta/2) d\theta - 4 \int_{\pi}^{2\pi} \cos(\theta/2) d\theta \\ &= 8 \sin(\theta/2) \Big|_{\theta=0}^{\theta=\pi} - 8 \sin(\theta/2) \Big|_{\theta=\pi}^{\theta=2\pi} = 16. \end{aligned}$$

3. Finally, $D_3 = \{(x, y, z) \in \mathbb{R}^3 : x \in [0, 1], y \in [0, 1-x], z \in [0, 1-y^2]\}$.

$$\begin{aligned} \iiint_{D_3} z dx dy dz &= \int_0^1 dx \int_0^{1-x} dy \int_0^{1-y^2} z dz = \frac{1}{2} \int_0^1 dx \int_0^{1-x} (1-y^2)^2 dy \\ &= \frac{1}{2} \int_0^1 \left(y - \frac{2}{3}y^3 + \frac{1}{5}y^5 \Big|_{y=0}^{y=1-x} \right) dx \\ &= \frac{1}{2} \int_0^1 \left((1-x) - \frac{2}{3}(1-x)^3 + \frac{1}{5}(1-x)^5 \right) dx \\ &= \frac{1}{2} \left(-\frac{1}{2}(1-x)^2 + \frac{1}{6}(1-x)^4 - \frac{1}{30}(1-x)^6 \right) \Big|_{x=0}^{x=1} = \frac{11}{60}. \end{aligned}$$

Exercise 5.

For the two domains D_1 and D_2 , we use cylindrical coordinates $x = r \cos \theta$, $y = r \sin \theta$, and $z = z$.

- The paraboloid, obtained by rotating around the Oz axis of the curve $z = r^2$, intersects the sphere, obtained by rotating around the Oz axis of the curve $r^2 + z^2 = 1$, when $z^2 + z - 1 = 0$ and $z > 0$, hence at $z^* = \frac{\sqrt{5}-1}{2}$. The domain $D_1 = P \cup B$ is therefore composed of a piece of paraboloid and a piece of sphere

$$\begin{aligned} P &= \{(r \cos \theta, r \sin \theta, z) \in \mathbb{R}^3 : z \in [0, z^*], \theta \in [0, 2\pi], r \in [0, \sqrt{z}]\}, \\ B &= \{(r \cos \theta, r \sin \theta, z) \in \mathbb{R}^3 : z \in [z^*, 1], \theta \in [0, 2\pi], r \in [0, \sqrt{1-z^2}]\}. \end{aligned}$$

Its volume is given by

$$\begin{aligned}
\text{Volume}(D_1) &= \text{Volume}(P) + \text{Volume}(B) = \iiint_P dx dy dz + \iiint_B dx dy dz \\
&= \int_0^{\frac{\sqrt{5}-1}{2}} dz \int_0^{2\pi} d\theta \int_0^{\sqrt{z}} r dr + \int_{\frac{\sqrt{5}-1}{2}}^1 dz \int_0^{2\pi} d\theta \int_0^{\sqrt{1-z^2}} r dr \\
&= 2\pi \int_0^{\frac{\sqrt{5}-1}{2}} \frac{z}{2} dz + 2\pi \int_{\frac{\sqrt{5}-1}{2}}^1 \frac{1-z^2}{2} dz = \frac{5\pi}{12} (3 - \sqrt{5}).
\end{aligned}$$

2. For the domain D_2 , the condition $x^2 + y^2 \leq 1$ translates to $0 \leq r \leq 1$. The two conditions $x \geq 0$ and $y \geq 0$ translate to $0 \leq \theta \leq \pi/2$. The last conditions $z \geq 0$ and $x + y + z \leq \sqrt{2}$ become $0 \leq z \leq \sqrt{2} - r \cos \theta - r \sin \theta$. Note, however, that for all $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi/2$, the quantity $\sqrt{2} - r \cos \theta - r \sin \theta \geq 0$, with equality if and only if $r = 1$ and $\theta = \pi/4$. Thus,

$$D_2 = \left\{ (r \cos \theta, r \sin \theta, z) \in \mathbb{R}^3 : r \in [0, 1], \theta \in [0, \pi/2], z \in [0, \sqrt{2} - r \cos \theta - r \sin \theta] \right\}.$$

Its volume is given by

$$\begin{aligned}
\text{Volume}(D_2) &= \iiint_{D_2} dx dy dz = \int_0^1 r dr \int_0^{\pi/2} d\theta \int_0^{\sqrt{2}-r \cos \theta - r \sin \theta} dz \\
&= \int_0^1 r dr \int_0^{\pi/2} (\sqrt{2} - r \cos \theta - r \sin \theta) d\theta \\
&= \int_0^1 r \left(\frac{\sqrt{2}\pi}{2} - r - r \right) dr = \frac{\sqrt{2}\pi}{4} - \frac{2}{3}.
\end{aligned}$$