

Exercise 1.**Remark.**

Since the domain of all our vector fields is \mathbb{R}^2 , which is simply connected, these fields derive from a potential if and only if $\text{rot } F = 0$ on \mathbb{R}^2 . Exceptionally, in the case where F does not derive from a potential, we are asked for a closed path such that $\int_{\Gamma} F \cdot dl \neq 0$. In general, choosing a random closed curve works

(if $\text{rot } F \neq 0$, there are very few closed curves for which $\int_{\Gamma} F \cdot dl = 0$.) We will see later (with Green's theorem) that if we choose a curve which is the boundary of a region Ω such that $\text{rot } F > 0$ on Ω (or $\text{rot } F < 0$ on Ω), then we have

$$\int_{\partial\Omega} F \cdot dl = \int_{\Omega} \text{rot } F dx dy > 0 \neq 0 \quad (\text{or } < 0 \neq 0)$$

1. We have

$$\text{rot } F(x, y) = \frac{\partial F_2}{\partial x}(x, y) - \frac{\partial F_1}{\partial y}(x, y) = \frac{\partial}{\partial x} [xy - x] - \frac{\partial}{\partial y} [y] = x - 1 - 1 = x - 2 \neq 0$$

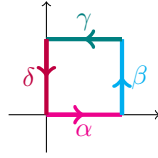
so F does not derive from a potential.

Let us choose Γ at random, for example $\Gamma = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, which we parametrize with $\gamma: [0, 2\pi] \rightarrow \mathbb{R}^2$ defined by $\gamma(t) = (\cos t, \sin t)$.

We then have

$$\begin{aligned} F(\gamma(t)) &= (\sin t, \cos t \sin t - \cos t) \\ \gamma'(t) &= (-\sin t, \cos t) \\ \int_{\Gamma} F \cdot dl &= \int_0^{2\pi} \langle (\sin t, \cos t \sin t - \cos t), (-\sin t, \cos t) \rangle dt = \int_0^{2\pi} -\sin^2 t - \cos t \sin^2 t - \cos^2 t dt \\ &= -\int_0^{2\pi} 1 + \sin^2 t \cos t dt = \left[t + \frac{1}{3} \sin^3 t \right]_0^{2\pi} = -2\pi \end{aligned}$$

Alternatively, we can choose the boundary of the square $[0, 1] \times [0, 1]$. The fact that our curve passes along the axes often gives us $x = 0$ or $y = 0$, which simplifies the calculation. Moreover, parametrizing line segments gives polynomial expressions, which we can always integrate (since the expression of F is also polynomial).



We parametrize these curves using the standard parametrization for a segment from $a = (a_1, a_2)$ to $b = (b_1, b_2)$: $t \mapsto tb + (1 - t)a = a + t(b - a)$, for $t \in [0, 1]$.

$$\alpha(t) = t(1, 0) + (1 - t)(0, 0) = (t, 0)$$

$$F(\alpha(t)) = (0, t \cdot 0 - t) = (0, -t)$$

$$\alpha'(t) = (1, 0)$$

$$\int_0^1 \langle F(\alpha(t)), \alpha'(t) \rangle dt = \int_0^1 \langle (0, -t), (1, 0) \rangle dt = \int_0^1 0 dt = 0$$

$$\beta(t) = t(1, 1) + (1 - t)(1, 0) = (1, t)$$

$$F(\beta(t)) = (t, t - 1)$$

$$\beta'(t) = (0, 1)$$

$$\int_0^1 \langle F(\beta(t)), \beta'(t) \rangle dt = \int_0^1 \langle (t, t - 1), (0, 1) \rangle dt = \int_0^1 t - 1 dt = \left[\frac{1}{2}t^2 - t \right]_0^1 = -\frac{1}{2}$$

$$\gamma(t) = t(0, 1) + (1 - t)(1, 1) = (1 - t, 1)$$

$$F(\gamma(t)) = (1, (1 - t) - (1 - t)) = (1, 0)$$

$$\gamma'(t) = (-1, 0)$$

$$\int_0^1 \langle F(\gamma(t)), \gamma'(t) \rangle dt = \int_0^1 \langle (1, 0), (-1, 0) \rangle dt = \int_0^1 -1 dt = -1$$

$$\delta(t) = t(0, 0) + (1 - t)(0, 1) = (0, 1 - t)$$

$$F(\delta(t)) = (1 - t, (1 - t) \cdot 0 - 0) = (1 - t, 0)$$

$$\delta'(t) = (0, -1)$$

$$\int_0^1 \langle F(\delta(t)), \delta'(t) \rangle dt = \int_0^1 \langle (1 - t, 0), (0, -1) \rangle dt = \int_0^1 0 dt = 0$$

To conclude,

$$\int_{\Gamma} F \cdot dl = \int_0^1 \langle F(\alpha(t)), \alpha'(t) \rangle dt + \int_0^1 \langle F(\beta(t)), \beta'(t) \rangle dt \\ + \int_0^1 \langle F(\gamma(t)), \gamma'(t) \rangle dt + \int_0^1 \langle F(\delta(t)), \delta'(t) \rangle dt = -\frac{3}{2} \neq 0$$

2. We have

$$\text{rot } F(x, y) = \frac{\partial F_2}{\partial x}(x, y) - \frac{\partial F_1}{\partial y}(x, y) = \frac{\partial}{\partial x} [x^3] - \frac{\partial}{\partial y} [3x^2y + 2x] = 3x^2 - 3x^2 = 0$$

and therefore F derives from a potential. We must then find $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\frac{\partial f}{\partial x}(x, y) = F_1(x, y) = 3x^2y + 2x \\ \frac{\partial f}{\partial y}(x, y) = F_2(x, y) = x^3$$

We can choose to either start by integrating the first equation with respect to x or start by integrating the second equation with respect to y .

Option 1: Start by integrating the first equation with respect to x .

We have

$$f(x, y) = \int \frac{\partial f}{\partial x}(x, y) dx = \int 3x^2y + 2x dx = x^3y + x^2 + C(y),$$

where, since we integrated with respect to x , our constant may depend on y . We substitute this into our second equation:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [x^3y + x^2 + C(y)] = x^3 + C'(y) := x^3.$$

Thus, we must have $C'(y) = 0$, meaning that C is a constant. Finally, the potentials of F are

$$f(x, y) = x^3y + x^2 + C,$$

with $C \in \mathbb{R}$. (There is no need to keep the constant; giving one potential is enough.)

Option 2: Start by integrating the second equation with respect to y .

We have

$$f(x, y) = \int \frac{\partial f}{\partial y}(x, y) dy = \int x^3 dy = x^3y + C(x),$$

where, since we integrated with respect to y , our constant may depend on x . We substitute this into our first equation:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} [x^3 y + C(x)] = 3x^2 y + C'(x) := 3x^2 y + 2x.$$

Thus, we must have $C'(x) = 2x$, so $C(x) = x^2 + K$ where K is a constant. Finally, the potentials of F are

$$f(x, y) = x^3 y + x^2 + K,$$

with $K \in \mathbb{R}$. (There is no need to keep the constant; giving one potential is enough.)

3. We have

$$\text{rot } F(x, y) = \frac{\partial F_2}{\partial x}(x, y) - \frac{\partial F_1}{\partial y}(x, y) = \frac{\partial}{\partial x} [x^2] - \frac{\partial}{\partial y} [3x^2 y] = 2x - 3x^2 \neq 0.$$

so F does not derive from a potential.

Let us choose Γ at random, for example $\Gamma = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ parametrized with $\gamma: [0, 2\pi] \rightarrow \mathbb{R}^2$ defined by $\gamma(t) = (\cos(t), \sin(t))$. We then have

$$\begin{aligned} F(\gamma(t)) &= (3 \cos^2 t \sin t, \cos^2 t) \\ \gamma'(t) &= (-\sin t, \cos t) \\ \int_{\Gamma} F \cdot dl &= \int_0^{2\pi} \langle (3 \cos^2 t \sin t, \cos^2 t), (-\sin t, \cos t) \rangle dt = \int_0^{2\pi} -3 \cos^2 t \sin^2 t + \cos^3 t dt. \end{aligned}$$

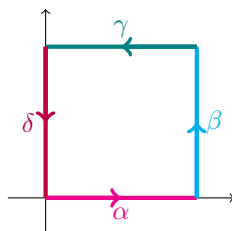
We compute the antiderivatives using the technique developed in Series 1 with Euler's formulas:

$$\begin{aligned} \cos^2 t \sin^2 t &= \left(\frac{e^{it} + e^{-it}}{2} \right)^2 \left(\frac{e^{it} - e^{-it}}{2i} \right)^2 = \frac{-1}{16} (e^{2it} + 2 + e^{-2it}) (e^{2it} - 2 + e^{-2it}) \\ &= \frac{-1}{16} (e^{4it} - 2e^{2it} + 1 + 2e^{2it} - 4 + 2e^{-2it} + 1 - e^{-2it} + e^{-4it}) \\ &= \frac{-1}{16} (e^{4it} + e^{-4it} - 2) = -\frac{1}{8} (\cos 4t - 1) \\ \cos^3 t &= \left(\frac{e^{it} + e^{-it}}{2} \right)^3 = \frac{1}{8} (e^{3it} + e^{-3it} + 3e^{it} + 3e^{-it}) \\ &= \frac{1}{4} \cos(3t) + \frac{3}{4} \cos(t), \end{aligned}$$

thus,

$$\begin{aligned} \int_{\Gamma} F \cdot dl &= \int_0^{2\pi} \left[\frac{3}{8} \cos 4t + \frac{3}{8} + \frac{1}{4} \cos(3t) + \frac{3}{4} \cos(t) \right] dt \\ &= \left[\frac{3}{32} \sin(4t) + \frac{3}{8} t + \frac{1}{12} \sin(3t) + \frac{3}{4} \sin(t) \right]_0^{2\pi} = \frac{3\pi}{4} \neq 0. \end{aligned}$$

Alternatively, as in point 1, we could take the boundary of the square $[0, 2] \times [0, 2]$ (if we take the boundary of the square $[0, 1] \times [0, 1]$, we unfortunately get 0. Bad luck...)



We then have,

$$\begin{aligned}
 \alpha(t) &= (2t, 0) \\
 F(\alpha(t)) &= (3(2t)^2 \cdot 0, (2t)^2) = (0, 4t^2) \\
 \alpha'(t) &= (2, 0) \\
 \int_0^1 \langle F(\alpha(t)), \alpha'(t) \rangle dt &= \int_0^1 \langle (0, 4t^2), (2, 0) \rangle dt = \int_0^1 0 dt = 0 \\
 \\
 \beta(t) &= (2, 2t) \\
 F(\beta(t)) &= (24t, 4) \\
 \beta'(t) &= (0, 2) \\
 \int_0^1 \langle F(\beta(t)), \beta'(t) \rangle dt &= \int_0^1 \langle (24t, 4), (0, 2) \rangle dt = \int_0^1 8 dt = 8 \\
 \\
 \gamma(t) &= (2 - 2t, 2) \\
 F(\gamma(t)) &= (24(1 - t)^2, 4(1 - t)^2) \\
 \gamma'(t) &= (-2, 0) \\
 \int_0^1 \langle F(\gamma(t)), \gamma'(t) \rangle dt &= \int_0^1 \langle (24(1 - t)^2, 4(1 - t)^2), (-2, 0) \rangle dt \\
 &= \int_0^1 -48(1 - t)^2 dt = [16(1 - t)^3]_0^1 = -16 \\
 \\
 \delta(t) &= (0, 2 - 2t) \\
 F(\delta(t)) &= (3 \cdot 0 \cdot (2 - 2t), 0) = (0, 0) \\
 \delta'(t) &= (0, -2) \\
 \int_0^1 \langle F(\delta(t)), \delta'(t) \rangle dt &= \int_0^1 \langle (0, 0), (0, -2) \rangle dt = \int_0^1 0 dt = 0.
 \end{aligned}$$

Finally,

$$\begin{aligned} \int_{\Gamma} F \cdot dl &= \int_0^1 \langle F(\alpha(t)), \alpha'(t) \rangle dt + \int_0^1 \langle F(\beta(t)), \beta'(t) \rangle dt \\ &\quad + \int_0^1 \langle F(\gamma(t)), \gamma'(t) \rangle dt + \int_0^1 \langle F(\delta(t)), \delta'(t) \rangle dt = -8 \neq 0 \end{aligned}$$

Exercise 2.

We have

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y} \left[2xy + \frac{z}{1+x^2} \right] = 2x \\ \frac{\partial F_1}{\partial z} &= \frac{\partial}{\partial z} \left[2xy + \frac{z}{1+x^2} \right] = \frac{1}{1+x^2} \\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x} [x^2 + 2yz] = 2x \\ \frac{\partial F_2}{\partial z} &= \frac{\partial}{\partial z} [x^2 + 2yz] = 2y \\ \frac{\partial F_3}{\partial x} &= \frac{\partial}{\partial x} [y^2 + \arctan x] = \frac{1}{1+x^2} \\ \frac{\partial F_3}{\partial y} &= \frac{\partial}{\partial y} [y^2 + \arctan x] = 2y \end{aligned}$$

$$\text{rot } F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = (0, 0, 0)$$

Moreover, since the domain of definition of F is \mathbb{R}^3 , which is simply connected, F derives from a potential. We therefore seek f such that

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial x} = F_1 = 2xy + \frac{z}{1+x^2} \\ \frac{\partial f}{\partial y} = F_2 = x^2 + 2yz \\ \frac{\partial f}{\partial z} = F_3 = y^2 + \arctan x \end{array} \right.$$

Option 1: Integrate the first equation with respect to x .

We have

$$f(x, y, z) = \int \frac{\partial f}{\partial x}(x, y, z) dx = \int 2xy + \frac{z}{1+x^2} dx = x^2y + z \arctan x + C(y, z),$$

where, since we integrated with respect to x , the constant may depend on y and z .

Option 1.1: Substitute into the second equation and integrate with respect to y .

We have

$$\begin{aligned}\frac{\partial f}{\partial y}(x, y, z) &= \frac{\partial}{\partial y} [x^2y + z \arctan x + C(y, z)] = x^2 + \frac{\partial C}{\partial y}(y, z) := x^2 + 2yz \\ \Rightarrow \frac{\partial C}{\partial y}(y, z) &= 2yz \\ C(y, z) &= \int \frac{\partial C}{\partial y}(y, z) dy = \int 2yz dy = y^2z + K(z), \\ \Rightarrow f(x, y, z) &= x^2y + y^2z + z \arctan x + K(z)\end{aligned}$$

where, since we integrated with respect to y , the constant K may depend on z (but not on x since C does not depend on x).

Finally, substituting this into the third equation and integrating with respect to z , we have

$$\begin{aligned}\frac{\partial f}{\partial z}(x, y, z) &= \frac{\partial}{\partial z} [x^2y + y^2z + z \arctan x + K(z)] = y^2 + \arctan x + K'(z) := y^2 + \arctan x \\ \Rightarrow K'(z) &= 0 \\ K(z) &= \text{const}\end{aligned}$$

and finally,

$$f(x, y, z) = x^2y + y^2z + z \arctan x + K,$$

with $K \in \mathbb{R}$. (No need to keep the constant; giving one potential is enough.)

Option 1.2: Substitute into the third equation and integrate with respect to z .

We have

$$\begin{aligned}\frac{\partial f}{\partial z}(x, y, z) &= \frac{\partial}{\partial z} [x^2y + z \arctan x + C(y, z)] = \arctan x + \frac{\partial C}{\partial z}(y, z) := y^2 + \arctan x \\ \Rightarrow \frac{\partial C}{\partial z}(y, z) &= y^2 \\ C(y, z) &= \int \frac{\partial C}{\partial z}(y, z) dz = y^2z + K(y) \\ \Rightarrow f(x, y, z) &= x^2y + y^2z + z \arctan x + K(y)\end{aligned}$$

where, since we integrated with respect to z , the constant K may depend on y (but not on x since C does not depend on x).

Finally, substituting this into the second equation and integrating with respect

to y , we have

$$\begin{aligned}\frac{\partial f}{\partial y}(x, y, z) &= \frac{\partial}{\partial y} [x^2y + y^2z + z \arctan x + K(y)] = x^2 + 2yz + K'(y) := x^2 + 2yz \\ \Rightarrow K'(y) &= 0 \\ K(y) &= \text{const}\end{aligned}$$

and finally

$$f(x, y, z) = x^2y + y^2z + z \arctan x + K,$$

with $K \in \mathbb{R}$. (No need to keep the constant; giving one potential is enough.)

Option 2: Integrate the second equation with respect to y .

We have

$$f(x, y, z) = \int \frac{\partial f}{\partial y}(x, y, z) dy = \int x^2 + 2yz dy = x^2y + y^2z + C(x, z)$$

where, since we integrated with respect to y , the constant may depend on x and z .

Option 2.1: Substitute into the first equation and integrate with respect to x .

We have

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y, z) &= \frac{\partial}{\partial x} [x^2y + y^2z + C(x, z)] = 2xy + \frac{\partial C}{\partial x}(x, z) := 2xy + \frac{z}{1+x^2} \\ \Rightarrow \frac{\partial C}{\partial x}(x, z) &= \frac{z}{1+x^2} \\ C(x, z) &= \int \frac{\partial C}{\partial x}(x, z) dx = \int \frac{z}{1+x^2} dx = z \arctan x + K(z) \\ \Rightarrow f(x, y, z) &= x^2y + y^2z + z \arctan x + K(z)\end{aligned}$$

where, since we integrated with respect to x , the constant K may depend on z (but not on y since C does not depend on y).

Finally, substituting this into the third equation and integrating with respect to z , we have

$$\begin{aligned}\frac{\partial f}{\partial z}(x, y, z) &= \frac{\partial}{\partial z} [x^2y + y^2z + z \arctan x + K(z)] = y^2 + \arctan x + K'(z) := y^2 + \arctan x \\ \Rightarrow K'(z) &= 0 \\ K(z) &= \text{const}\end{aligned}$$

and finally,

$$f(x, y, z) = x^2y + y^2z + z \arctan x + K,$$

with $K \in \mathbb{R}$. (No need to keep the constant; giving one potential is enough.)

Option 2.2: Substitute into the third equation and integrate with respect to z .

We have

$$\begin{aligned}\frac{\partial f}{\partial z}(x, y, z) &= \frac{\partial}{\partial z} [x^2y + y^2z + C(x, z)] = y^2 + \frac{\partial C}{\partial z}(x, z) := y^2 + \arctan x \\ \Rightarrow \frac{\partial C}{\partial z}(x, z) &= \arctan x \\ C(x, z) &= \int \frac{\partial C}{\partial z}(x, z) dz = \int \arctan x dz = z \arctan x + K(x) \\ \Rightarrow f(x, y, z) &= x^2y + y^2z + z \arctan x + K(x)\end{aligned}$$

where, since we integrated with respect to z , the constant K may depend on x (but not on y since C does not depend on y).

Finally, substituting this into the first equation and integrating with respect to x , we have

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y, z) &= \frac{\partial}{\partial x} [x^2y + y^2z + z \arctan x + K(x)] = 2xy + \frac{z}{1+x^2} := 2xy + \frac{z}{1+x^2} \\ \Rightarrow K'(x) &= 0 \\ K(x) &= \text{const}\end{aligned}$$

and finally,

$$f(x, y, z) = x^2y + y^2z + z \arctan x + K,$$

with $K \in \mathbb{R}$. (No need to keep the constant; giving one potential is enough.)

Option 3: Integrate the third equation with respect to z .

We have

$$f(x, y, z) = \int \frac{\partial f}{\partial z}(x, y, z) dz = \int y^2 + \arctan x dz = y^2z + z \arctan x + C(x, y),$$

where, since we integrated with respect to z , the constant C may depend on x and y .

Option 3.1: Substitute into the first equation and integrate with respect to x .

We have

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y, z) &= \frac{\partial}{\partial x} [y^2z + z \arctan x + C(x, y)] = \frac{z}{1+x^2} + \frac{\partial C}{\partial x}(x, y) := 2xy + \frac{z}{1+x^2} \\ \Rightarrow \frac{\partial C}{\partial x}(x, y) &= 2xy \\ C(x, y) &= \int \frac{\partial C}{\partial x}(x, y) dx = \int 2xy dx = x^2y + K(y) \\ \Rightarrow f(x, y, z) &= x^2y + y^2z + z \arctan x + K(y)\end{aligned}$$

where, since we integrated with respect to x , the constant K may depend on y (but not on z since C does not depend on z).

Finally, substituting this into the second equation and integrating with respect to y , we have

$$\begin{aligned}\frac{\partial f}{\partial y}(x, y, z) &= \frac{\partial}{\partial y} [x^2y + y^2z + z \arctan x + K(y)] = x^2 + 2yz + K'(y) := x^2 + 2yz \\ \Rightarrow K'(y) &= 0 \\ K(y) &= \text{const}\end{aligned}$$

and finally

$$f(x, y, z) = x^2y + y^2z + z \arctan x + K,$$

with $K \in \mathbb{R}$. (No need to keep the constant; giving one potential is enough.)

Option 3.2: Substitute into the second equation and integrate with respect to y .

We have

$$\begin{aligned}\frac{\partial f}{\partial y}(x, y, z) &= \frac{\partial}{\partial y} [y^2z + z \arctan x + C(x, y)] = 2yz + \frac{\partial C}{\partial y}(x, y) := x^2 + 2yz \\ \Rightarrow \frac{\partial C}{\partial y}(x, y) &= x^2 \\ C(x, y) &= \int \frac{\partial C}{\partial y}(x, y) dy = \int x^2 dy = x^2y + K(x) \\ \Rightarrow f(x, y, z) &= x^2y + y^2z + z \arctan x + K(x)\end{aligned}$$

where, since we integrated with respect to y , the constant K may depend on x (but not on z since C does not depend on z).

Finally, substituting this into the first equation and integrating with respect to x , we have

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y, z) &= \frac{\partial}{\partial x} \left[x^2y + y^2z + \frac{z}{1+x^2} + K(x) \right] = 2xy + \frac{z}{1+x^2} := 2xy + \frac{z}{1+x^2} \\ \Rightarrow K'(x) &= 0 \\ K(x) &= \text{const}\end{aligned}$$

and finally,

$$f(x, y, z) = x^2y + y^2z + z \arctan x + K,$$

with $K \in \mathbb{R}$. (No need to keep the constant; giving one potential is enough.)

Exercise 3.

For the vector fields $F: \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}^2$ below, do they derive from a potential? If yes, give a potential; if not, justify your answer.

1. $F(x, y) = \left(\frac{-x}{(x^2 + y^2)^2}, \frac{-y}{(x^2 + y^2)^2} \right)$

$$2. F(x, y) = \left(\frac{y^3}{(x^2 + y^2)^2}, \frac{-xy^2}{(x^2 + y^2)^2} \right)$$

We use the method from the course. **In this solution, whenever we have to choose between looking for a potential and looking for a closed path, we will systematically make the wrong choice to see how to detect that we went in the wrong direction.**

1. Step 1: Compute $\text{rot } F$.

We have

$$\begin{aligned} \text{rot } F(x, y) &= \frac{\partial F_2}{\partial x}(x, y) - \frac{\partial F_1}{\partial y}(x, y) = \frac{\partial}{\partial x} \left[\frac{-y}{(x^2 + y^2)^2} \right] - \frac{\partial}{\partial y} \left[\frac{-x}{(x^2 + y^2)^2} \right] \\ &= \left(-y(-2) \frac{\partial}{\partial x} [x^2 + y^2] \right) - \left(-x(-2) \frac{\partial}{\partial y} [x^2 + y^2] \right) \\ &= 4y \frac{x}{(x^2 + y^2)^3} - \left(4x \frac{y}{(x^2 + y^2)^3} \right) = 0 \end{aligned}$$

The curl of F is zero, so we move to

Step 2: Domain of F .

The domain of F is $\mathbb{R}^2 \setminus \{(0, 0)\}$, which is not simply connected; it has a hole at the origin. So we move to step 3, for which we have a choice: try to find a potential, or try to compute the integral over a closed path that surrounds the hole in the domain. Here, the field will derive from a potential (there's no real way to know this at this stage); for pedagogical reasons, we first test trying to find a curve that surrounds a hole in the domain.

Step 3: For each hole in the domain, integrate F over a closed path that surrounds that hole.

Here, the only hole in the domain is the origin, so we only have one path to find, and we choose $\Gamma = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, parameterized by $\gamma: [0, 2\pi] \rightarrow \mathbb{R}^2$ defined by $\gamma(t) = (\cos t, \sin t)$.

We then have

$$\begin{aligned} F(\gamma(t)) &= (-\cos t, -\sin t) \\ \gamma'(t) &= (-\sin t, \cos t) \\ \int_{\gamma} F \cdot dl &= \int_0^{2\pi} \langle (-\cos t, -\sin t), (-\sin t, \cos t) \rangle dt = \int_0^{2\pi} \cos t \sin t - \cos t \sin t dt = \int_0^{2\pi} 0 dt = 0 \end{aligned}$$

Since there is only one hole in the domain and the integral over a closed contour around this hole gives 0, we suspect that the integral over all closed paths is zero, i.e., that F derives from a potential. We therefore change our approach: we will look for a potential for F .

Step 4: We look for a potential.

We seek $f \in C^1(\mathbb{R}^2 \setminus \{(0, 0)\})$ such that

$$\begin{cases} \frac{\partial f}{\partial x}(x, y) = \frac{-x}{(x^2 + y^2)^2} \\ \frac{\partial f}{\partial y}(x, y) = \frac{-y}{(x^2 + y^2)^2} \end{cases}$$

Since our expressions are symmetric in x and y , we choose to start by integrating the first equation with respect to x . The calculations in the case where we start by integrating the second equation with respect to y are the same; we just swap the roles of x and y .

We have

$$f(x, y) = \int \frac{\partial f}{\partial x}(x, y) dx = \int \frac{-x}{(x^2 + y^2)^2} dx = \frac{1}{2(x^2 + y^2)} + C(y),$$

where, since we integrated with respect to x , the constant is allowed to depend on y .

Injecting into the second equation gives

$$\begin{aligned} \frac{\partial f}{\partial y}(x, y) &= \frac{\partial}{\partial y} \left[\frac{1}{2(x^2 + y^2)} + C(y) \right] = \frac{-y}{2(x^2 + y^2)^2} + C'(y) := \frac{-y}{2(x^2 + y^2)^2} \\ &\Rightarrow C'(y) = 0 \\ &C(y) = \text{const.} \end{aligned}$$

Finally, the potentials of F are

$$f(x, y) = \frac{1}{2(x^2 + y^2)} + C,$$

with $C \in \mathbb{R}$.

2. Step 1: Compute $\text{rot } F$.

We have

$$\begin{aligned}
\text{rot } F(x, y) &= \frac{\partial F_2}{\partial x}(x, y) - \frac{\partial F_1}{\partial y}(x, y) = \frac{\partial}{\partial x} \left[\frac{-xy^2}{(x^2 + y^2)^2} \right] - \frac{\partial}{\partial y} \left[\frac{y^3}{(x^2 + y^2)^2} \right] \\
&= \frac{-y^2(x^2 + y^2)^2 + xy^2 \frac{\partial}{\partial x} [(x^2 + y^2)^2]}{(x^2 + y^2)^4} - \frac{3y^2(x^2 + y^2)^2 - y^3 \frac{\partial}{\partial y} [(x^2 + y^2)^2]}{(x^2 + y^2)^4} \\
&= \frac{-y^2(x^2 + y^2)^2 + xy^2 \cdot 4x(x^2 + y^2)}{(x^2 + y^2)^4} - \frac{3y^2(x^2 + y^2)^2 - y^3 \cdot 4y(x^2 + y^2)}{(x^2 + y^2)^4} \\
&= \frac{-y^2(x^2 + y^2) + 4x^2y^2 - 3y^2(x^2 + y^2) + 4y^4}{(x^2 + y^2)^3} \\
&= \frac{-y^2x^2 - y^4 + 4x^2y^2 - 3y^2x^2 - 3y^4 + 4y^4}{(x^2 + y^2)^3} = 0
\end{aligned}$$

The curl of F is zero, so we move to

Step 2: Domain of F .

The domain of F is $\mathbb{R}^2 \setminus \{(0, 0)\}$, which is not simply connected; it has a hole at the origin. We therefore move to Step 3, where we have a choice: either try to find a potential, or try to compute the integral over a closed path that surrounds the hole in the domain. Here, the field will not derive from a potential (there is no real way to know this at this stage), but for pedagogical reasons we first attempt to find a potential.

Step 3: We look for a potential for F .

We seek $f \in C^1(\mathbb{R}^2 \setminus \{(0, 0)\})$ such that

$$\begin{cases} \frac{\partial f}{\partial x}(x, y) = \frac{y^3}{(x^2 + y^2)^2} \\ \frac{\partial f}{\partial y}(x, y) = \frac{-xy^2}{(x^2 + y^2)^2} \end{cases}$$

We integrate the first equation with respect to x . (We could integrate the second equation with respect to y ; the calculations are not significantly easier, but not trivial either.)

We have

$$f(x, y) = \int \frac{\partial f}{\partial x}(x, y) dx = \int \frac{y^3}{(x^2 + y^2)^2} dx = \int \frac{y^{-4}}{y^{-4}} \frac{y^3}{(x^2 + y^2)^2} dx = \int \frac{1}{y} \frac{1}{\left(1 + \frac{x^2}{y^2}\right)^2} dx.$$

At this stage, it is not unreasonable that the awkward form of our integral makes us reconsider and either try integrating the second equation or find a closed path over which the integral of F does not vanish.

Indeed, the antiderivative of $\frac{1}{(1+t^2)^2}$ is not part of the standard Analysis I curriculum. However, this antiderivative is useful in the generalization of the method of decomposition into partial fractions when, in the factorization of the denominator, we have terms of the form $(x^2 + bx + c)^2$ where the polynomial $x^2 + bx + c$ is irreducible, i.e., it has no real roots.

Here we propose a method to calculate it (this **is not** the kind of thing you will need to know for the exam).

$$\begin{aligned} \int \frac{1}{(1+x^2)^2} dx &= \int \frac{1+x^2}{(1+x^2)^2} - \frac{x^2}{(1+x^2)^2} dx = \int \frac{1}{1+x^2} dx + \int \frac{x}{2} \frac{-2x}{(1+x^2)^2} dx \\ &\stackrel{\text{IBP}}{=} \arctan(x) + \frac{x}{2} \frac{1}{1+x^2} - \int \frac{1}{2} \frac{1}{1+x^2} dx \\ &= \frac{1}{2} \frac{x}{1+x^2} + \arctan(x) - \frac{1}{2} \arctan(x) \\ &= \frac{1}{2} \left(\frac{x}{1+x^2} + \arctan(x) \right). \end{aligned}$$

Continuing our calculation,

$$\begin{aligned} f(x, y) &= \int \frac{1}{y} \frac{1}{\left(1 + \left(\frac{x}{y}\right)^2\right)^2} dx = \frac{1}{2} \left(\frac{\frac{x}{y}}{1 + \frac{x^2}{y^2}} + \arctan\left(\frac{x}{y}\right) \right) + C(y) \\ &= \frac{1}{2} \frac{xy}{x^2 + y^2} + \frac{1}{2} \arctan\left(\frac{x}{y}\right) + C(y). \end{aligned}$$

At this stage, notice that the term $\arctan\left(\frac{x}{y}\right)$ does not have the same domain as F . Even without knowing $C(y)$ yet, the fact that $C(y)$ cannot depend on x , while $\arctan\left(\frac{x}{y}\right)$ depends on x , indicates that it is very unlikely that the exact expression of $C(y)$ could "fix" the domain issue with $\arctan(x/y)$. Specifically, this means that if, for example, $C(y) = -\frac{1}{2} \arctan\left(\frac{x}{y}\right)$ were admissible, then f would have no domain problem. But such a function $C(y)$ is not allowed because $\arctan(x/y)$ depends on x , which is not permitted for $C(y)$.

This is a hint that we have probably taken the wrong approach. But let's persevere, just out of morbid curiosity.

We inject this into the second equation:

$$\begin{aligned}
\frac{\partial f}{\partial y}(x, y) &= \frac{\partial}{\partial y} \left[\frac{1}{2} \frac{xy}{x^2 + y^2} + \frac{1}{2} \arctan\left(\frac{x}{y}\right) + C(y) \right] \\
&= \frac{1}{2} \frac{x(x^2 + y^2) - 2y \cdot xy}{(x^2 + y^2)^2} + \frac{1}{2} \frac{1}{1 + \frac{x^2}{y^2}} \frac{-x}{y^2} + C'(y) \\
&= \frac{1}{2} \frac{x^3 + xy^2 - 2xy^2}{(x^2 + y^2)^2} - \frac{1}{2} \frac{x^3 + xy^2}{(x^2 + y^2)^2} + C'(y) \\
&= \frac{-xy^2}{(x^2 + y^2)^2} + C'(y) \\
\Rightarrow C'(y) &= 0 \\
C(y) &= \text{const.}
\end{aligned}$$

Finally,

$$f(x, y) = \frac{1}{2} \frac{xy}{x^2 + y^2} + \frac{1}{2} \arctan\left(\frac{x}{y}\right) + C,$$

with $C \in \mathbb{R}$.

As already mentioned, we cannot choose a constant such that $f \in C^1(\mathbb{R}^2 \setminus \{(0, 0)\})$, so we change approach and look for a closed curve on which the integral does not vanish.

Step 4: For each hole in the domain, we look for a closed curve surrounding the hole such that the integral of F over this curve does not vanish.

Since $(0, 0)$ is the only hole in the domain, we choose $\Gamma = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ parametrized by $\gamma: [0, 2\pi] \rightarrow \mathbb{R}^2$ defined by $\gamma(t) = (\cos t, \sin t)$.

We then have

$$\begin{aligned}
F(\gamma(t)) &= (\sin^3 t, -\cos t \sin^2 t) \\
\gamma'(t) &= (-\sin t, \cos t) \\
\int_{\Gamma} F \cdot dl &= \int_0^{2\pi} \langle F(\gamma(t)), \gamma'(t) \rangle dt = \int_0^{2\pi} -\sin^4 t - \cos^2 t \sin^2 t dt \\
&= \int_0^{2\pi} -\sin^2 t (\sin^2 t + \cos^2 t) dt = -\int_0^{2\pi} \sin^2 t dt.
\end{aligned}$$

We find an antiderivative of $\sin^2 t$ using Euler's formula:

$$\sin^2 t = \left(\frac{e^{it} - e^{-it}}{2i} \right)^2 = -\frac{1}{4} (e^{2it} - 2 + e^{-2it}) = \frac{1}{2} - \frac{1}{2} \cos(2t)$$

and therefore,

$$\int_{\Gamma} F \cdot dl = \int_0^{2\pi} \frac{1}{2} \cos(2t) - \frac{1}{2} dt = \left[\frac{1}{4} \sin(2t) - \frac{t}{2} \right]_0^{2\pi} = -\pi \neq 0$$

and thus F does not derive from a potential.

Exercise 4.

1. Given that the vector field $F(x, y) = (F_1(x, y), F_2(x, y))$ derives from a potential f in \mathbb{R}^2 , we know:

$$F_1(x, y) = \frac{\partial f(x, y)}{\partial x} \quad \text{and} \quad F_2(x, y) = \frac{\partial f(x, y)}{\partial y},$$

or equivalently

$$F_1(t, u(t)) = \frac{\partial f(t, u(t))}{\partial t} \quad \text{and} \quad F_2(t, u(t)) = \frac{\partial f(t, u(t))}{\partial u}.$$

Notice that the second derivative (respect to u) is just a partial derivative of F_2 respect its second variable.

To show that $f(t, u(t))$ is constant, we need to calculate its total derivative with respect to t :

$$\frac{d}{dt}f(t, u(t)) = \frac{\partial f}{\partial t}(t, u(t)) + u'(t)\frac{\partial f}{\partial u}(t, u(t)).$$

Here, we substitute:

$$\frac{d}{dt}f(t, u(t)) = F_1(t, u(t)) + F_2(t, u(t))u'(t).$$

From the differential equation:

$$F_2(t, u(t))u'(t) + F_1(t, u(t)) = 0,$$

therefore,

$$\frac{d}{dt}f(t, u(t)) = 0.$$

Since the derivative is equal to zero, $f(t, u(t))$ is constant for all $t \in \mathbb{R}$.

Thus, we have shown that a solution $u(t)$, in implicit form, of the differential equation is given by:

$$f(t, u(t)) = \text{constant}$$

for all $t \in \mathbb{R}$.

2. Given the equation:

$$u^2(t)u'(t) + \sin t = 0,$$

we can rewrite it as follows

$$F_2(t, u(t))u'(t) + F_1(t, u(t)) = 0,$$

where $F_1 = \sin t$ and $F_2 = u^2(t)$.

Since

$$\frac{d}{dt}f(t, u(t)) = F_1(t, u(t)) + F_2(t, u(t))u'(t),$$

and the fact that F is defined for $t \in \mathbb{R}$ we can compute f by taking an integral:

$$f(t, u(t)) = -\cos t + \frac{u^3(t)}{3}.$$

We just have shown that a solution $u(t)$, in implicit form, of the differential equation is given by:

$$f(t, u(t)) = \text{constant}.$$

Therefore

$$-\cos t + \frac{u^3(t)}{3} = c = \text{constant}.$$

Given the initial condition $u(0) = 3$, we can find the constant c :

$$-\cos 0 + \frac{u^3(0)}{3} = -1 + 9 = 8 = c.$$

Finally, combining the two last equations we get the solution:

$$u(t) = (3 \cos t + 24)^{1/3}.$$

Exercise 5.

For $i = 1, 2$, we have

$$\begin{aligned} \frac{d}{dt} [tF_i(tx, ty)] &= F_1(tx, ty) + t \left(\frac{\partial F_i}{\partial x}(tx, ty) \frac{d}{dt} [tx] + \frac{\partial F_i}{\partial y}(tx, ty) \frac{d}{dt} [ty] \right) \\ &= F_1(tx, ty) + tx \frac{\partial F_i}{\partial x}(tx, ty) + ty \frac{\partial F_i}{\partial y}(tx, ty). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{\partial \varphi}{\partial x}(x, y) &= \frac{\partial}{\partial x} \left[\int_0^1 xF_1(tx, ty) + yF_2(tx, ty) dt \right] \\ &= \int_0^1 \frac{\partial}{\partial x} [xF_1(tx, ty) + yF_2(tx, ty)] dt \\ &= \int_0^1 F_1(tx, ty) + x \left(\frac{\partial F_1}{\partial x}(tx, ty) \frac{\partial}{\partial x} [tx] + \frac{\partial F_1}{\partial y}(tx, ty) \frac{\partial}{\partial x} [ty] \right) \\ &\quad + y \left(\frac{\partial F_2}{\partial x}(tx, ty) \frac{\partial}{\partial x} [tx] + \frac{\partial F_2}{\partial y}(tx, ty) \frac{\partial}{\partial x} [ty] \right) dt \\ &= \int_0^1 F_1(tx, ty) + x \frac{\partial F_1}{\partial x}(tx, ty)t + y \frac{\partial F_2}{\partial x}(tx, ty)t dt \end{aligned}$$

Using that $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ and the calculation above, we obtain

$$\begin{aligned}\frac{\partial \varphi}{\partial x}(x, y) &= \int_0^1 F_1(tx, ty) + tx \frac{\partial F_1}{\partial x}(tx, ty) + ty \frac{\partial F_1}{\partial y}(tx, ty) dt \\ &= \int_0^1 \frac{d}{dt} [tF_1(tx, ty)] dt = [tF_1(tx, ty)]_{t=0}^{t=1} = F_1(x, y).\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{\partial \varphi}{\partial y}(x, y) &= \frac{\partial}{\partial y} \left[\int_0^1 xF_1(tx, ty) + yF_2(tx, ty) dt \right] \\ &= \int_0^1 \frac{\partial}{\partial y} [xF_1(tx, ty) + yF_2(tx, ty)] dt \\ &= \int_0^1 x \left(\frac{\partial F_1}{\partial x}(tx, ty) \frac{\partial}{\partial y} [tx] + \frac{\partial F_1}{\partial y}(tx, ty) \frac{\partial}{\partial y} [ty] \right) \\ &\quad + F_2(tx, ty) + y \left(\frac{\partial F_2}{\partial x}(tx, ty) \frac{\partial}{\partial y} [tx] + \frac{\partial F_2}{\partial y}(tx, ty) \frac{\partial}{\partial y} [ty] \right) dt \\ &= \int_0^1 F_2(tx, ty) + x \frac{\partial F_1}{\partial y}(tx, ty)t + y \frac{\partial F_2}{\partial y}(tx, ty)t dt\end{aligned}$$

Using that $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ and the calculation above, we obtain

$$\begin{aligned}\frac{\partial \varphi}{\partial x}(x, y) &= \int_0^1 F_2(tx, ty) + tx \frac{\partial F_2}{\partial x}(tx, ty) + ty \frac{\partial F_2}{\partial y}(tx, ty) dt \\ &= \int_0^1 \frac{d}{dt} [tF_2(tx, ty)] dt = [tF_2(tx, ty)]_{t=0}^{t=1} = F_2(x, y).\end{aligned}$$

Thus we have shown that φ is indeed a potential of F .

We set

$$\varphi(x, y) = \int_0^1 x \cdot 2 \cdot tx \cdot ty + y((tx)^2 + ty) dt = 3x^2y \int_0^1 t^2 dt + y^2 \int_0^1 t dt = x^2y + \frac{1}{2}y^2,$$

which, by the previous point, is a potential of F .

The formula from the course for an open set Ω star-shaped with respect to x_0 and a vector field $F: \Omega \rightarrow \mathbb{R}^n$ is

$$\varphi(x, y) = \int_0^1 \langle x - x_0, F(x_0 + t(x - x_0)) \rangle dt.$$

This is the same formula but with $x_0 = 0$.