EPFL - Fall 2025	Dr. Pablo Antolin
Analysis III (for IC) – MATH-203(d) – IN SC	**Exercises**
Correction 2	September 18 2025

Exercise 1.

1. Let's start with F_1. We have

$$\begin{aligned}\frac{\partial F_1}{\partial x}(x,y,z) =& y^2 z\cos(xz)\\ \frac{\partial F_1}{\partial y}(x,y,z) =& 2y\sin(xz)\\ \frac{\partial F_1}{\partial z}(x,y,z) =& xy^2\cos(xz)\end{aligned}$$

and therefore

$$\nabla F_1(x,y,z) = \big(y^2 z\cos(xz), 2y\sin(xz), xy^2\cos(xz)\big)$$

Let's move on to F_2. We have

$$\begin{aligned}\frac{\partial F_2}{\partial x}(x,y,z) =& -2xe^y\sin(x^2+z)\\ \frac{\partial F_2}{\partial y}(x,y,z) =& e^y\cos(x^2+z)\\ \frac{\partial F_2}{\partial z}(x,y,z) =& -e^y\sin(x^2+z)\end{aligned}$$

and therefore

$$\nabla F_2(x,y,z) = \big(-2xe^y\sin(x^2+z), e^y\cos(x^2+z), -e^y\sin(x^2+z)\big)$$

Let's finish with F_3. We have

$$\begin{aligned}\frac{\partial F_3}{\partial x}(x,y,z) =& \frac{\frac{\partial}{\partial x}\left[2+\cos(xy)\right]}{2+\cos(xy)} = \frac{-y\sin(xy)}{2+\cos(xy)}\\ \frac{\partial F_3}{\partial y}(x,y,z) =& \frac{\frac{\partial}{\partial y}\left[2+\cos(xy)\right]}{2+\cos(xy)} = \frac{-x\sin(xy)}{2+\cos(xy)}\\ \frac{\partial F_3}{\partial z}(x,y,z) =& 0\end{aligned}$$

and therefore

$$\nabla F_3(x,y,z) = \left(\frac{-y\sin(xy)}{2+\cos(xy)}, \frac{-x\sin(xy)}{2+\cos(xy)}, 0\right)$$

2. We have

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$
$$= y^2 z \cos(xz) + e^y \cos(x^2 + z) + 0$$

3. We have

$$\operatorname{rot} F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$
$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$
$$= \left(\frac{-x \sin(xy)}{2 + \cos(xy)} + e^y \sin(x^2 + z), xy^2 \cos(xz) - \frac{-y \sin(xy)}{2 + \cos(xy)}, -2xe^y \sin(x^2 + z) - 2y \sin(xz) \right)$$

Exercise 2.
The goal is to use the following rules:

- ∇ applies to scalar fields[1]
- ∇ gives a vector field
- rot and div apply to vector fields.
- rot gives a vector field (since we are in $\mathbb{R}^3$. If we were in $\mathbb{R}^2$ it would be a scalar field.)
- div gives a scalar field.

1. *OK*: ∇ is defined for scalar fields. We obtain a vector field.
2. *OK*: ∇f is defined and is a vector field. We can then multiply the vector field ∇f by the scalar field f: we use pointwise scalar multiplication of a vector.
3. *OK*: F and ∇f are vector fields, so we can take their dot product.
4. *Oh hell nah!*: The divergence only applies to vector fields; f is a scalar field.
5. *OK*: $f \cdot F$, obtained by scalar multiplication of the vector field F by f, is a vector field. The divergence indeed applies to vector fields.
6. *OK*: $f \cdot F$, obtained by scalar multiplication of the vector field F by f, is a vector field. The curl indeed applies to vector fields.

[1] Here we interpret ∇ as the *gradient*. In reality, ∇ also applies to functions $g\colon \mathbb{R}^m \to \mathbb{R}^n$ and is interpreted as the *Jacobian matrix*.

7. *Oh hell nah!*: The curl only applies to vector fields; f is a scalar field.
8. *OK*: F is a vector field and, since we are in $\mathbb{R}^3$, $\operatorname{rot} F$ is also a vector field. Then, we multiply by the scalar f.
9. *Oh hell nah!*: $\operatorname{div} F$ is a scalar field and the curl requires a vector field.

Exercise 3.
We have that f is the composition of several functions. Therefore, by proceeding step by step,

$$\frac{\partial f}{\partial x_j}(x) = \frac{\partial}{\partial x_j}\left[r(x)^{-1}\right] = -r(x)^{-2}\frac{\partial r}{\partial x_j}(x) = \frac{-1}{\sum_{i=1}^n (x_i - a_i)^2}\frac{\partial}{\partial x_j}\left[\sqrt{\sum_{i=1}^n (x_i - a_i)^2}\right]$$
$$= \frac{-1}{\sum_{i=1}^n (x_i - a_i)^2}\frac{1}{2}\frac{1}{\sqrt{\sum_{i=1}^n (x_i - a_i)^2}}\frac{\partial}{\partial x_j}\left[\sum_{i=1}^n (x_i - a_i)^2\right]$$
$$= \frac{-1}{2\left(\sum_{i=1}^n (x_i - a_i)^2\right)^{\frac{3}{2}}}\sum_{i=1}^n \frac{\partial}{\partial x_j}\left[(x_i - a_i)^2\right]$$

Note at this stage that if $i \neq j$, then $(x_i - a_i)^2$ does not depend on x_j, and therefore $\frac{\partial}{\partial x_j}\left[(x_i - a_i)^2\right] = 0$, whereas if $i = j$, $\frac{\partial}{\partial x_j}\left[(x_i - a_i)^2\right] = 2(x_j - a_j)$.

Thus,

$$\frac{\partial f}{\partial x_j}(x) = \frac{-1}{2\left(\sum_{i=1}^n (x_i - a_i)^2\right)^{\frac{3}{2}}}2(x_j - a_j) = \frac{a_j - x_j}{r(x)^3}$$

We move on to the second-order derivative,

$$\frac{\partial^2 f}{\partial x_j^2}(x) = \frac{\partial}{\partial x_j}\left[\frac{a_j - x_j}{r(x)^3}\right] = -\frac{1}{r(x)^3} + (a_j - x_j)\frac{\partial}{\partial x_j}\left[r(x)^{-3}\right] = -\frac{1}{r(x)^3} + (a_j - x_j)(-3)r(x)^{-4}\frac{\partial r}{\partial x_j}(x)$$
$$= -\frac{1}{r(x)^3} + (a_j - x_j)(-3)r(x)^{-4}\frac{\partial}{\partial x_j}\left[\sqrt{\sum_{i=1}^n (x_i - a_i)^2}\right]$$
$$= -\frac{1}{r(x)^3} + \frac{3(x_j - a_j)}{r(x)^4}\frac{1}{2}\frac{1}{\sqrt{\sum_{i=1}^n (x_i - a_i)^2}}2(x_j - a_j)$$
$$= -\frac{1}{r(x)^3} + 3\frac{(x_j - a_j)^2}{r(x)^5}.$$

Finally,

$$\begin{aligned}\Delta f(x) =& \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}(x)\\ =& \sum_{j=1}^{n}\left(-\frac{1}{r(x)^3}+3\frac{(x_j-a_j)^2}{r(x)^5}\right)\\ =& -\frac{n}{r(x)^3}+\frac{3}{r(x)^5}\underbrace{\sum_{j=1}^{n}(x_j-a_j)^2}_{=r(x)^2}\\ =& \frac{3-n}{r(x)^3}\end{aligned}$$

Exercise 4.
For the first three points, we carry out the demonstration in the more general case where $\Omega \subset \mathbb{R}^n$ with arbitrary n. For the last two, they are only valid for $n = 3$.

1. We have

$$\begin{aligned}\operatorname{div}(f\nabla g) =& \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[(f\nabla g)_j\right] = \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[f\frac{\partial g}{\partial x_j}\right] = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_j}+f\frac{\partial^2 g}{\partial x_j^2}\right)\\ =& \underbrace{\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_j}}_{=\langle\nabla f,\nabla g\rangle}+f\underbrace{\sum_{j=1}^{n}\frac{\partial^2 g}{\partial x_j^2}}_{=\Delta g},\end{aligned}$$

which is the desired result.

2. Since

$$\frac{\partial}{\partial x_j}[fg] = f\frac{\partial g}{\partial x_j}+g\frac{\partial f}{\partial x_j},$$

we have

$$\begin{aligned}\nabla(fg) =& \left(f\frac{\partial g}{\partial x_1}+g\frac{\partial f}{\partial x_1},...,f\frac{\partial g}{\partial x_n}+g\frac{\partial f}{\partial x_n}\right)\\ =& f\left(\frac{\partial g}{\partial x_1},...,\frac{\partial g}{\partial x_n}\right)+g\left(\frac{\partial f}{\partial x_1},...,\frac{\partial f}{\partial x_n}\right)\\ =& f\nabla g+g\nabla f.\end{aligned}$$

3. We have

$$\begin{aligned}\operatorname{div}(fF) &= \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[(fF)_j\right] = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[fF_j\right] = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j}F_j + f\frac{\partial F_j}{\partial x_j}\right)\\ &= \underbrace{\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}F_j}_{=\langle \nabla f, F\rangle} + f\underbrace{\sum_{j=1}^{n}\frac{\partial F_j}{\partial x_j}}_{=\operatorname{div} F},\end{aligned}$$

which is the desired result.

4. We have

$$\operatorname{rot} F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right).$$

Thus, (note that since F is C^2, when we take second-order partial derivatives, the order of the variables with respect to which we differentiate does not matter)

$$\begin{aligned}\operatorname{rot}\operatorname{rot} F &= \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}\\ &= \left(\frac{\partial^2 F_2}{\partial x \partial y} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial^2 z} + \frac{\partial^2 F_3}{\partial x \partial z}\right) e_1\\ &\quad + \left(-\frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial y \partial z} - \frac{\partial^2 F_2}{\partial z^2}\right) e_2\\ &\quad + \left(\frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y \partial z}\right) e_3.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{\partial}{\partial x}\left[\operatorname{div} F\right] &= \frac{\partial}{\partial x}\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right]\\ &= \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z}\\ \frac{\partial}{\partial y}\left[\operatorname{div} F\right] &= \frac{\partial}{\partial y}\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right]\\ &= \frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y \partial z}\\ \frac{\partial}{\partial z}\left[\operatorname{div} F\right] &= \frac{\partial}{\partial z}\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right]\\ &= \frac{\partial^2 F_1}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} + \frac{\partial^2 F_3}{\partial z^2}\end{aligned}$$

Thus,

$$\nabla \operatorname{div} F = \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z}, \frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y \partial z}, \frac{\partial^2 F_1}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} + \frac{\partial^2 F_3}{\partial z^2} \right)$$

and therefore

$$\begin{aligned} \nabla \operatorname{div} F - \operatorname{rot}\operatorname{rot} F = &\left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}, \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2}, \frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2} \right) \\ =&\Delta F. \end{aligned}$$

By rearranging the terms, we obtain the desired result.

5. We have

$$\begin{aligned} \operatorname{rot}(fF) = &\begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix} \\ =&\left(\frac{\partial}{\partial y}[fF_3] - \frac{\partial}{\partial z}[fF_2], \frac{\partial}{\partial z}[fF_1] - \frac{\partial}{\partial x}[fF_3], \frac{\partial}{\partial x}[fF_2] - \frac{\partial}{\partial y}[fF_1] \right) \\ =&f \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \\ &+ \left(\frac{\partial f}{\partial y} F_3 - \frac{\partial f}{\partial z} F_2, \frac{\partial f}{\partial z} F_1 - \frac{\partial f}{\partial x} F_3, \frac{\partial f}{\partial x} F_2 - \frac{\partial f}{\partial y} F_1 \right) \\ =&r \operatorname{rot} F + \left(\frac{\partial f}{\partial y} F_3 - \frac{\partial f}{\partial z} F_2, \frac{\partial f}{\partial z} F_1 - \frac{\partial f}{\partial x} F_3, \frac{\partial f}{\partial x} F_2 - \frac{\partial f}{\partial y} F_1 \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \nabla f \wedge F = &\begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ =&\left(\frac{\partial f}{\partial y} F_3 - \frac{\partial f}{\partial z} F_2, \frac{\partial f}{\partial z} F_1 - \frac{\partial f}{\partial x} F_3, \frac{\partial f}{\partial x} F_2 - \frac{\partial f}{\partial y} F_1 \right). \end{aligned}$$

By comparing with what we obtained in the calculation of $\operatorname{rot}(fF)$, we indeed get

$$\operatorname{rot}(fF) = \nabla f \wedge F + f \operatorname{rot} F,$$

which is the desired result.

Exercise 5. 1. <u>*Version 1*</u> : Direct calculation.

We have

$$
\begin{aligned}
\frac{\partial g}{\partial r}(r,\theta) =& \frac{\partial}{\partial r}\left[f(r\cos\theta, r\sin\theta)\right] \\
=& \frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial r}\left[r\cos\theta\right] + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial r}\left[r\sin\theta\right] \\
=& \frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\cos\theta + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\sin\theta \\
\frac{\partial^2 g}{\partial r^2}(r,\theta) =& \frac{\partial}{\partial r}\left[\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\cos\theta + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\sin\theta\right] \\
=& \frac{\partial}{\partial r}\left[\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\right]\cos\theta + \frac{\partial}{\partial r}\left[\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\right]\sin\theta \\
=& \left(\frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial r}\left[r\cos\theta\right] + \frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial r}\left[r\sin\theta\right]\right)\cos\theta \\
&+ \left(\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial r}\left[r\cos\theta\right] + \frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial r}\left[r\sin\theta\right]\right)\sin\theta \\
=& \frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta)\cos^2\theta + 2\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta)\cos\theta\sin\theta \\
&+ \frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta)\sin^2\theta \\
\frac{\partial g}{\partial \theta}(r,\theta) =& \frac{\partial}{\partial \theta}\left[f(r\cos\theta, r\sin\theta)\right] \\
=& \frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial \theta}\left[r\cos\theta\right] + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial \theta}\left[r\sin\theta\right] \\
=& r\left(-\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\sin\theta + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\cos\theta\right) \\
\frac{\partial^2 g}{\partial \theta^2}(r,\theta) =& \frac{\partial}{\partial \theta}\left[r\left(-\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\sin\theta + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\cos\theta\right)\right] \\
=& r\left(-\frac{\partial}{\partial \theta}\left[\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\right]\sin\theta - \frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\cos\theta\right. \\
&\quad\left.+\frac{\partial}{\partial \theta}\left[\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\right]\cos\theta - \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\sin\theta\right) \\
=& -r\sin\theta\left(\frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial \theta}\left[r\cos\theta\right] + \frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial \theta}\left[r\sin\theta\right]\right) \\
&- r\cos\theta\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) \\
&+ r\cos\theta\left(\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial \theta}\left[r\cos\theta\right] + \frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial \theta}\left[r\sin\theta\right]\right) \\
&- r\sin\theta\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta) \\
=& r^2\sin^2\theta\frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta) - 2r^2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta) \\
&+ r^2\cos^2\theta\frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta) \\
&- r\cos\theta\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) - r\sin\theta\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)
\end{aligned}
$$

Thus,

$$\begin{aligned}
\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r}+\frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}& \\
&= \underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{=1}\frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta) + \underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{=1}\frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta) \\
&= \Delta f(r\cos\theta, r\sin\theta)
\end{aligned}$$

<u>*Variante 2*</u> : With physicists' tricks.

If $x(r,\theta) = r\cos\theta$ and $y(r,\theta) = r\sin\theta$, we have

$$\begin{aligned}
\frac{\partial x}{\partial r} &= \cos\theta = \frac{x}{r} \\
\frac{\partial x}{\partial \theta} &= -r\sin\theta = -y \\
\frac{\partial y}{\partial r} &= \sin\theta = \frac{y}{r} \\
\frac{\partial y}{\partial \theta} &= r\cos\theta = x
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial g}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{1}{r}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) \\
\frac{\partial^2 g}{\partial r^2} &= -\frac{1}{r^2}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) \\
&\quad + \frac{1}{r}\left(\frac{\partial x}{\partial r}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial f}{\partial y} + x\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial x\partial y}\frac{\partial y}{\partial r}\right) + y\left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial r}\right)\right) \\
&= -\frac{1}{r^2}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) \\
&\quad + \frac{1}{r}\left(\frac{x}{r}\frac{\partial f}{\partial x} + \frac{y}{r}\frac{\partial f}{\partial y} + x\left(\frac{\partial^2 f}{\partial x^2}\frac{x}{r} + \frac{\partial^2 f}{\partial x\partial y}\frac{y}{r}\right) + y\left(\frac{\partial^2 f}{\partial x\partial y}\frac{x}{r} + \frac{\partial^2 f}{\partial y^2}\frac{y}{r}\right)\right) \\
&= \frac{1}{r^2}\left(x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right) \\
\frac{\partial g}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} \\
\frac{\partial^2 g}{\partial \theta^2} &= -\frac{\partial y}{\partial \theta}\frac{\partial f}{\partial x} - y\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial x\partial y}\frac{\partial y}{\partial \theta}\right) + \frac{\partial x}{\partial \theta}\frac{\partial f}{\partial y} + x\left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial \theta}\right) \\
&= -x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} - y\left(-y\frac{\partial^2 f}{\partial x^2} + x\frac{\partial^2 f}{\partial x\partial y}\right) + x\left(-y\frac{\partial^2 f}{\partial x\partial y} + x\frac{\partial^2 f}{\partial y^2}\right) \\
&= -x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} + y^2\frac{\partial^2 f}{\partial x^2} - 2xy\frac{\partial^2 f}{\partial x\partial y} + x^2\frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

And therefore we conclude

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} = \underbrace{\frac{x^2+y^2}{r^2}}_{=1}\frac{\partial^2 f}{\partial x^2} + \underbrace{\frac{x^2+y^2}{r^2}}_{=1}\frac{\partial^2 f}{\partial y^2} = \Delta f$$

2. Let us define

$$g(r,\theta) = f(r\cos\theta, r\sin\theta) = r + \left(\arctan\frac{\sin\theta}{\cos\theta}\right)^2 = r + (\arctan\tan)^2 = r + \theta^2$$

Then,

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} = 0 + \frac{1}{r}\cdot 1 + \frac{1}{r^2}\cdot 2 = \frac{2+r}{r^2} = \Delta f(r\cos\theta, r\sin\theta).$$

Thus,

$$\Delta f(x,y) = \frac{2+\sqrt{x^2+y^2}}{x^2+y^2}$$