

Exercise 1.

1. We have:

$$\begin{aligned}
 \frac{\partial}{\partial x_i} [\langle \mathbf{F}, \mathbf{G} \rangle] &= \frac{\partial}{\partial x_i} \left[\sum_{j=1}^n F_j G_j \right] \\
 &= \sum_{j=1}^n \frac{\partial}{\partial x_i} [F_j G_j] \\
 &= \sum_{j=1}^n \left(\frac{\partial F_j}{\partial x_i} G_j + F_j \frac{\partial G_j}{\partial x_i} \right) \\
 &= \sum_{j=1}^n \frac{\partial F_j}{\partial x_i} G_j + \sum_{j=1}^n F_j \frac{\partial G_j}{\partial x_i},
 \end{aligned}$$

which leads to:

$$\frac{\partial}{\partial x_i} [\langle \mathbf{F}, \mathbf{G} \rangle] = \left\langle \frac{\partial \mathbf{F}}{\partial x_i}, \mathbf{G} \right\rangle + \left\langle \mathbf{F}, \frac{\partial \mathbf{G}}{\partial x_i} \right\rangle$$

2. Let us consider the first component. We use the subscript $(\cdot)_i$ to denote the i -th component of a given vector. We have:

$$\begin{aligned}
 \left(\frac{\partial}{\partial x_i} [\mathbf{F} \wedge \mathbf{G}] \right)_1 &= \frac{\partial}{\partial x_i} [(\mathbf{F} \wedge \mathbf{G})_1] \\
 &= \frac{\partial}{\partial x_i} [F_2 G_3 - F_3 G_2] \\
 &= \frac{\partial}{\partial x_i} [F_2 G_3] - \frac{\partial}{\partial x_i} [F_3 G_2] \\
 &= \frac{\partial F_2}{\partial x_i} G_3 + F_2 \frac{\partial G_3}{\partial x_i} - \left(\frac{\partial F_3}{\partial x_i} G_2 + F_3 \frac{\partial G_2}{\partial x_i} \right) \\
 &= \frac{\partial F_2}{\partial x_i} G_3 - \frac{\partial F_3}{\partial x_i} G_2 + F_2 \frac{\partial G_3}{\partial x_i} - F_3 \frac{\partial G_2}{\partial x_i} \\
 &= \left(\frac{\partial \mathbf{F}}{\partial x_i} \right)_2 G_3 - \left(\frac{\partial \mathbf{F}}{\partial x_i} \right)_3 G_2 + F_2 \left(\frac{\partial \mathbf{G}}{\partial x_i} \right)_3 - F_3 \left(\frac{\partial \mathbf{G}}{\partial x_i} \right)_2 \\
 &= \left(\frac{\partial \mathbf{F}}{\partial x_i} \wedge \mathbf{G} \right)_1 + \left(\mathbf{F} \wedge \frac{\partial \mathbf{G}}{\partial x_i} \right)_1 \\
 &= \left(\frac{\partial \mathbf{F}}{\partial x_i} \wedge \mathbf{G} + \mathbf{F} \wedge \frac{\partial \mathbf{G}}{\partial x_i} \right)_1
 \end{aligned}$$

The exact same procedure holds true for the second and third components. Therefore, we show that

$$\frac{\partial}{\partial x_i} [\mathbf{F} \wedge \mathbf{G}] = \frac{\partial \mathbf{F}}{\partial x_i} \wedge \mathbf{G} + \mathbf{F} \wedge \frac{\partial \mathbf{G}}{\partial x_i}.$$

Exercise 2.

1. Polar coordinates:

$$\text{Jac}(u)(r, \theta) = \det \nabla u(r, \theta) = \det \begin{pmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{pmatrix} = r.$$

2. Spherical coordinates:

$$\begin{aligned} \text{Jac}(u)(r, \theta, \varphi) &= \det \nabla u(r, \theta, \varphi) \\ &= \det \begin{pmatrix} \cos(\theta) \sin(\varphi) & -r \sin(\theta) \sin(\varphi) & r \cos(\theta) \cos(\varphi) \\ \sin(\theta) \sin(\varphi) & r \cos(\theta) \sin(\varphi) & r \sin(\theta) \cos(\varphi) \\ \cos(\varphi) & 0 & -r \sin(\varphi) \end{pmatrix} \\ &= -r^2 \sin(\varphi). \end{aligned}$$

3. Cylindrical coordinates:

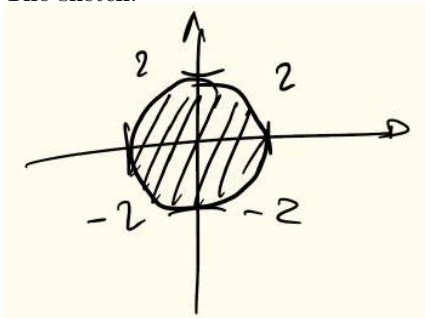
$$\text{Jac}(u)(r, \theta, z) = \det \nabla u(r, \theta, z) = \det \begin{pmatrix} \cos(\theta) & -r \sin(\theta) & 0 \\ \sin(\theta) & r \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = r.$$

4. Cartesian coordinates:

$$\text{Jac}(u)(x, y, z) = \det \nabla u(x, y, z) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1.$$

Exercise 3.

1. The sketch:



The term $x^2 + y^2$ in the description of A suggests that it is appropriate to use polar coordinates: $(x, y) = (r \cos \theta, r \sin \theta)$ with $r \geq 0$ and $\theta \in [0, 2\pi]$,

$$x^2 + y^2 \leq 4 \iff r^2 \leq 4 \stackrel{r \geq 0}{\iff} 0 \leq r \leq 2.$$

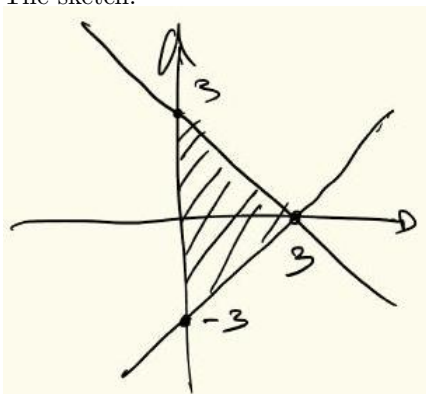
Thus,

$$A = \{(r \cos \theta, r \sin \theta) \mid r \in [0, 2], \theta \in [0, 2\pi]\}$$

and we compute

$$\begin{aligned} \int_A f(y) dy &= \int_0^2 \int_0^{2\pi} f(r \cos \theta, r \sin \theta) r d\theta dr \\ &= \int_0^2 \int_0^{2\pi} \frac{1}{r} r d\theta dr \\ &= 4\pi. \end{aligned}$$

2. The sketch:



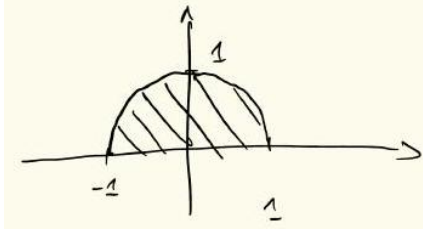
Here, we can use a Cartesian coordinate system directly:

$$\begin{aligned} \int_A f(\mathbf{x}) d\mathbf{x} &= \int_0^3 \int_{x-3}^{3-x} f(x, y) dy dx \\ &= \int_0^3 \int_{x-3}^{3-x} x^2 + \sin^3(y) dy dx \\ &= \int_0^3 (6 - 2x)x^2 dx + \underbrace{\int_0^3 \int_{x-3}^{3-x} \sin^3(y) dy dx}_{=0}. \end{aligned}$$

The second integral is zero because the integrand is an odd function and the integration domain is symmetric with respect to the origin. Thus:

$$\begin{aligned} \int_A f(\mathbf{x}) d\mathbf{x} &= \int_0^3 6x^2 - 2x^3 dx \\ &= \left[2x^3 - \frac{1}{2}x^4 \right]_{x=0}^{x=3} \\ &= \frac{27}{2}. \end{aligned}$$

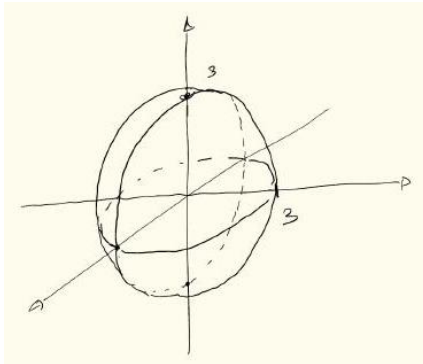
3. The sketch:



We can either use a Cartesian or a Polat coordinate system here. With Cartesian coordinates, we get:

$$\begin{aligned}
 \int_A f(x) dx &= \int_{-1}^1 \int_0^{\sqrt{1-x^2}} y(1+x^2) dy dx \\
 &= \int_{-1}^1 (1+x^2) \left[\frac{1}{2} y^2 \right]_{y=0}^{y=\sqrt{1-x^2}} dx \\
 &= \frac{1}{2} \int_{-1}^1 (1+x^2)(1-x^2) dx \\
 &= \frac{1}{2} \int_{-1}^1 -x^4 + 1 dx \\
 &= \frac{1}{2} \left[-\frac{1}{5} x^5 + x \right]_{x=-1}^{x=1} = \frac{4}{5}.
 \end{aligned}$$

4. The sketch:



The term $x^2 + y^2 + z^2$ in the description of A suggests that it is appropriate to use spherical coordinates: $(x, y, z) = (r \cos \theta \sin \varphi, r \sin \theta \sin \varphi, r \cos \varphi)$ with $r \geq 0$, $\theta \in [0, 2\pi]$, and $\varphi \in [0, \pi]$,

$$x^2 + y^2 + z^2 \leq 9 \iff r^2 \leq 9 \stackrel{r \geq 0}{\iff} 0 \leq r \leq 3.$$

Thus,

$$A = \{(r \cos \theta \sin \varphi, r \sin \theta \sin \varphi, r \cos \varphi) \mid r \in [0, 3], \theta \in [0, 2\pi], \varphi \in [0, \pi]\}$$

and we compute

$$\begin{aligned} \int_A f(y) dy &= \int_0^3 \int_0^{2\pi} \int_0^\pi r \sin \varphi r^2 \sin \varphi d\varphi d\theta dr \\ &= 2\pi \int_0^3 r^3 dr \int_0^\pi \sin^2 \varphi d\varphi. \end{aligned}$$

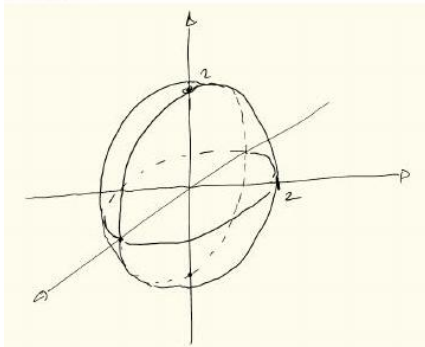
We recall that

$$\frac{d}{dt} \left[\frac{1}{2} (t - \sin(t) \cos(t)) \right] = \sin^2(t).$$

Therefore,

$$\begin{aligned} \int_A f(y) dy &= 2\pi \left[\frac{1}{4} r^4 \right]_{r=0}^{r=3} \left[\frac{1}{2} (\varphi - \sin(\varphi) \cos(\varphi)) \right]_{\varphi=0}^{\varphi=\pi} \\ &= \frac{81}{4} \pi^2. \end{aligned}$$

5. The sketch:



We employ the spherical coordinate system which leads to define the domain of integration as:

$$A = \{(r \cos \theta \sin \varphi, r \sin \theta \sin \varphi, r \cos \varphi) \mid r \in [0, 2], \theta \in [0, 2\pi], \varphi \in [0, \pi]\}.$$

We obtain:

$$\begin{aligned} \int_A f(\mathbf{x}) d\mathbf{x} &= \int_0^2 \int_0^{2\pi} \int_0^\pi \arccos(\cos \varphi) r^2 \sin \varphi d\varphi d\theta dr \\ &= 2\pi \int_0^2 r^2 dr \int_0^\pi \varphi \sin \varphi d\varphi \\ &= 2\pi \left[\frac{1}{3} r^3 \right]_{r=0}^{r=2} \left([\varphi(-\cos \varphi)]_{\varphi=0}^{\varphi=\pi} + \int_0^\pi \cos \varphi d\varphi \right) \\ &= \frac{16}{3} \pi^2. \end{aligned}$$

Exercise 4.

1. In a first step, we can show that:

$$\nabla(|x|) = \frac{x}{|x|}, \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

A straightforward computation shows:

$$\frac{\partial}{\partial x_i} |x| = \frac{\partial}{\partial x_i} \left[\sqrt{\sum_{j=1}^n x_j^2} \right] = \frac{1}{2\sqrt{\sum_{j=1}^n x_j^2}} \frac{\partial}{\partial x_i} \left[\sum_{j=1}^n x_j^2 \right] = \frac{1}{2|x|} \sum_{j=1}^n \underbrace{\frac{\partial}{\partial x_i} [x_j^2]}_{=0 \text{ if } i \neq j \text{ and } =2x_i \text{ if } i=j} = \frac{x_i}{|x|}.$$

Consequently, we have:

$$\nabla(|x|^p) = p|x|^{p-1} \nabla(|x|) = p|x|^{p-2} x.$$

Remark: this function is well defined at $x = 0$ if $p \geq 2$.

2. By defining,

$$\begin{aligned} h_p : \mathbb{R}^n &\rightarrow \mathbb{R} : u \mapsto h_p(u) \\ G : \mathbb{R}^n &\rightarrow \mathbb{R}^n : v \mapsto G(v) \\ \gamma : \mathbb{R} &\rightarrow \mathbb{R}^n : t \mapsto \gamma(t) = tx \end{aligned}$$

we can rewrite the function f_p as a composition of these functions:

$$f_p = \frac{1}{p} h_p \circ G \circ \gamma.$$

Therefore, by applying the chain rule of differentiation, we have:

$$\begin{aligned} \frac{d}{dt} f_p(t) &= \frac{1}{p} \sum_{i=1}^n \frac{\partial h_p}{\partial u_i} (G(\gamma(t))) \frac{dG_i \circ \gamma}{dt}(t) \\ &= \frac{1}{p} \sum_{i=1}^n \frac{\partial h_p}{\partial u_i} (G(\gamma(t))) \sum_{j=1}^n \frac{\partial G_i}{\partial v_j} (\gamma(t)) \frac{d\gamma^j}{dt}(t) \end{aligned}$$

Since,

$$\frac{\partial h_p}{\partial u_i} (u) = p|u|^{p-2} u_i, \quad \text{and} \quad \frac{d\gamma^j}{dt}(t) = x_j,$$

we get the final result:

$$\frac{d}{dt} f_p(t) = \sum_{i=1}^n \sum_{j=1}^n |G(tx)|^{p-2} G_i(tx) \frac{\partial G_i}{\partial v_j}(tx) x_j.$$