EPFL – Automne 2025	D. Strütt
Analyse III – CGC GC SIE	**Exercices**
Série 2	18 septembre 2025

Remarque.
Certains exercices proviennent du livre *Analyse Avancée pour Ingénieurs*, par B. Dacorogna et C. Tanteri.

Exercice 1 (ex 1.1, p. 7, corrigé p. 8)**.**
Soit

$$F(x,y,z) = \Big(y^2\sin(xz), e^y\cos(x^2+z), \log(2+\cos(xy))\Big) = (F_1, F_2, F_3).$$

Calculer :

(*i*) $\operatorname{grad} F_1$, $\operatorname{grad} F_2$, $\operatorname{grad} F_3$

(*ii*) $\operatorname{div} F$

(*iii*) $\operatorname{rot} F$.

Solution :

(*i*) Commençons par F_1. On a

$$\begin{aligned}\frac{\partial F_1}{\partial x}(x,y,z) =& y^2 z\cos(xz)\\ \frac{\partial F_1}{\partial y}(x,y,z) =& 2y\sin(xz)\\ \frac{\partial F_1}{\partial z}(x,y,z) =& xy^2\cos(xz)\end{aligned}$$

et donc

$$\nabla F_1(x,y,z) = \Big(y^2 z\cos(xz), 2y\sin(xz), xy^2\cos(xz)\Big)$$

Passons à F_2. On a

$$\begin{aligned}\frac{\partial F_2}{\partial x}(x,y,z) =& -2xe^y\sin(x^2+z)\\ \frac{\partial F_2}{\partial y}(x,y,z) =& e^y\cos(x^2+z)\\ \frac{\partial F_2}{\partial z}(x,y,z) =& -e^y\sin(x^2+z)\end{aligned}$$

et donc

$$\nabla F_2(x,y,z) = \Big(-2xe^y\sin(x^2+z), e^y\cos(x^2+z), -e^y\sin(x^2+z)\Big)$$

Finissons par F_3. On a

$$\begin{aligned}\frac{\partial F_3}{\partial x}(x,y,z) =& \frac{\frac{\partial}{\partial x}[2+\cos(xy)]}{2+\cos(xy)} = \frac{-y\sin(xy)}{2+\cos(xy)}\\ \frac{\partial F_3}{\partial y}(x,y,z) =& \frac{\frac{\partial}{\partial y}[2+\cos(xy)]}{2+\cos(xy)} = \frac{-x\sin(xy)}{2+\cos(xy)}\\ \frac{\partial F_3}{\partial z}(x,y,z) =& 0\end{aligned}$$

et donc

$$\nabla F_3(x,y,z) = \left(\frac{-y\sin(xy)}{2+\cos(xy)}, \frac{-x\sin(xy)}{2+\cos(xy)}, 0\right)$$

(*ii*) On a

$$\begin{aligned}\operatorname{div} F =& \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ =& y^2 z\cos(xz) + e^y\cos(x^2+z) + 0\end{aligned}$$

(*iii*) On a

$$\begin{aligned}\operatorname{rot} F &= \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\ &= \left(\frac{-x\sin(xy)}{2+\cos(xy)} + e^y\sin(x^2+z), xy^2\cos(xz) - \frac{-y\sin(xy)}{2+\cos(xy)}, -2xe^y\sin(x^2+z) - 2y\sin(xz)\right)\end{aligned}$$

Exercice 2 (ex 1.2, p. 7, corrigé p. 8)**.**
Si $f : \mathbb{R}^3 \to \mathbb{R}$ est $C^1(\mathbb{R}^3)$ et $F : \mathbb{R}^3 \to \mathbb{R}^3$ est $C^1(\mathbb{R}^3;\mathbb{R}^3)$, alors parmi les expressions suivantes lesquelles ont un sens ?

(*i*) ∇f
(*ii*) $f \cdot \nabla f$
(*iii*) $\langle F, \nabla f\rangle$
(*iv*) $\operatorname{div} f$
(*v*) $\operatorname{div}(f \cdot F)$
(*vi*) $\operatorname{rot}(f \cdot F)$
(*vii*) $\operatorname{rot} f$
(*viii*) $f \cdot \operatorname{rot} F$
(*ix*) $\operatorname{rot}\operatorname{div} F$

Solution :
Le but est d'utiliser les règles suivantes :

- ∇ s'applique à des champs scalaires[1]
- ∇ donne un champ vectoriel
- rot et div d'appliquent à des champs vectoriels.
- rot donne un champ vectoriel (vu qu'on est dans $\mathbb{R}^3$. Si on était dans $\mathbb{R}^2$ ça serait un champ scalaire.)
- div donne un champ scalaire.

(*i*) *OK* : ∇ est défini pour les champs scalaires. On obtient un champ vectoriel.

(*ii*) *OK* : ∇f est défini et est un champ vectoriel. On peut ensuite multiplier le champ vectoriel ∇f par le champ scalaire f : On utilise point par point la multiplication par scalaire d'un vecteur.

(*iii*) *OK* : F et ∇f sont des champs vectoriels et on peut donc prendre le produit scalaire des deux.

(*iv*) *Oh hell nah !* : La divergence ne s'applique qu'à des champs vectoriels ; f est un champ scalaire.

(*v*) *OK* : $f \cdot F$ obtenu par multiplication par scalaire du champ vectoriel F par f est un champ vectoriel. La divergence prend bien des champs vectoriels.

(*vi*) *OK* : $f \cdot F$ obtenu par multiplication par scalaire du champ vectoriel F par f est un champ vectoriel. Le rotationnel prend bien des champs vectoriels.

(*vii*) *Oh hell nah !* : Le rotationnel ne s'applique qu'à des champs vectoriels ; f est un champ scalaire.

(*viii*) *OK* : F est un champ scalaire et vu qu'on est dans $\mathbb{R}^3$, $\operatorname{rot} F$ est également un champ vectoriel. Ensuite, on multiplie par le scalaire f.

(*ix*) *Oh hell nah !* : $\operatorname{div} F$ est un champ scalaire et le rotationnel a besoin d'un champ vectoriel.

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[1] On interprète ici ∇ comme le *gradient.* En vrai, ∇ s'applique aussi à des fonction $g \colon \mathbb{R}^m \to \mathbb{R}^n$ et s'interprète comme la *matrice jacobienne*

Exercice 3 (exemple 1.3, p. 6)**.**
Soient $x = (x_1, \ldots, x_n)$, $a = (a_1, \ldots, a_n)$, et $r\colon \mathbb{R}^n \to \mathbb{R}$ définie par $r(x) = \|x - a\| = \sqrt{\sum_{i=1}^n (x_i - a_i)^2}$.
Soit f le champ scalaire défini par $f(x) = 1/r(x)$. Calculer Δf.
Solution :
On a que f est la composition de plusieurs fonctions. Donc, en procédant étape par étape,

$$\begin{aligned}\frac{\partial f}{\partial x_j}(x) =& \frac{\partial}{\partial x_j}\left[r(x)^{-1}\right] = -r(x)^{-2}\frac{\partial r}{\partial x_j}(x) = \frac{-1}{\sum_{i=1}^n (x_i - a_i)^2}\frac{\partial}{\partial x_j}\left[\sqrt{\sum_{i=1}^n (x_i - a_i)^2}\right]\\ =& \frac{-1}{\sum_{i=1}^n (x_i - a_i)^2}\frac{1}{2}\frac{1}{\sqrt{\sum_{i=1}^n (x_i - a_i)^2}}\frac{\partial}{\partial x_j}\left[\sum_{i=1}^n (x_i - a_i)^2\right]\\ =& \frac{-1}{2\left(\sum_{i=1}^n (x_i - a_i)^2\right)^{\frac{3}{2}}}\sum_{i=1}^n \frac{\partial}{\partial x_j}\left[(x_i - a_i)^2\right]\end{aligned}$$

Remarquons à ce stade que si $i \neq j$, alors $(x_i - a_i)^2$ ne dépend pas de x_j et donc $\frac{\partial}{\partial x_j}\left[(x_i - a_i)^2\right] = 0$, tandis que si $i = j$, $\frac{\partial}{\partial x_j}\left[(x_i - a_i)^2\right] = 2(x_j - a_j)$.
Ainsi,

$$\frac{\partial f}{\partial x_j}(x) = \frac{-1}{2\left(\sum_{i=1}^n (x_i - a_i)^2\right)^{\frac{3}{2}}} 2(x_j - a_j) = \frac{a_j - x_j}{r(x)^3}$$

On passe à la dérivée d'ordre 2,

$$\begin{aligned}\frac{\partial^2 f}{\partial x_j^2}(x) =& \frac{\partial}{\partial x_j}\left[\frac{a_j - x_j}{r(x)^3}\right] = -\frac{1}{r(x)^3} + (a_j - x_j)\frac{\partial}{\partial x_j}\left[r(x)^{-3}\right] = -\frac{1}{r(x)^3} + (a_j - x_j)(-3)r(x)^{-4}\frac{\partial r}{\partial x_j}(x)\\ =& -\frac{1}{r(x)^3} + (a_j - x_j)(-3)r(x)^{-4}\frac{\partial}{\partial x_j}\left[\sqrt{\sum_{i=1}^n (x_i - a_i)^2}\right]\\ =& -\frac{1}{r(x)^3} + \frac{3(x_j - a_j)}{r(x)^4}\frac{1}{2}\frac{1}{\sqrt{\sum_{i=1}^n (x_i - a_i)^2}} 2(x_j - a_j)\\ =& -\frac{1}{r(x)^3} + 3\frac{(x_j - a_j)^2}{r(x)^5}.\end{aligned}$$

Pour finir,

$$\begin{aligned}\Delta f(x) =& \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}(x)\\ =& \sum_{j=1}^n \left(-\frac{1}{r(x)^3} + 3\frac{(x_j - a_j)^2}{r(x)^5}\right)\\ =& -\frac{n}{r(x)^3} + \frac{3}{r(x)^5}\underbrace{\sum_{j=1}^n (x_j - a_j)^2}_{=r(x)^2}\\ =& \frac{3 - n}{r(x)^3}\end{aligned}$$

Exercice 4 (ex 1.6&1.7, p. 8, corrigé p. 11)**.**
Soit $\Omega \subset \mathbb{R}^3$ un ouvert. Montrer que :

(i) Si $f \in C^1(\Omega)$ et $g \in C^2(\Omega)$, alors :

$$\operatorname{div}(f \operatorname{grad} g) = f\Delta g + \langle \operatorname{grad} f, \operatorname{grad} g\rangle$$

(*ii*) Si $f, g \in C^1(\Omega)$, alors :

$$\operatorname{grad}(fg) = f \operatorname{grad} g + g \operatorname{grad} f$$

(*iii*) Si $f \in C^1(\Omega)$ et $F \in C^1(\Omega, \mathbb{R}^3)$ alors :

$$\operatorname{div}(fF) = f \operatorname{div} F + \langle F, \operatorname{grad} f\rangle$$

(*iv*) Si $F \in C^2(\Omega, \mathbb{R}^3)$, alors :

$$\operatorname{rot} \operatorname{rot} F = -\Delta F + \operatorname{grad} \operatorname{div} F,$$

où pour $F = (F_1, F_2, F_3)$ on note $\Delta F = (\Delta F_1, \Delta F_2, \Delta F_3)$.

(*v*) Si $f \in C^1(\Omega)$ et $F \in C^1(\Omega, \mathbb{R}^3)$, alors :

$$\operatorname{rot}(fF) = \operatorname{grad} f \wedge F + f \operatorname{rot} F$$

Solution :
Pour les 3 premiers points, on fait la démonstration dans le cas plus général où $\Omega \subset \mathbb{R}^n$ avec n quelconque. Pour les deux derniers, ils ne sont vrais que pour $n = 3$.

(*i*) On a

$$\begin{aligned}\operatorname{div}(f\nabla g) =&\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[(f\nabla g)_j\right] = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[f\frac{\partial g}{\partial x_j}\right] = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_j} + f\frac{\partial^2 g}{\partial x_j^2}\right)\\ =&\underbrace{\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_j}}_{=\langle\nabla f, \nabla g\rangle} + f\underbrace{\sum_{j=1}^{n}\frac{\partial^2 g}{\partial x_j^2}}_{=\Delta g},\end{aligned}$$

qui est le résultat voulu.

(*ii*) Vu que

$$\frac{\partial}{\partial x_j}[fg] = f\frac{\partial g}{\partial x_j} + g\frac{\partial f}{\partial x_j},$$

on a

$$\begin{aligned}\nabla(fg) =&\left(f\frac{\partial g}{\partial x_1} + g\frac{\partial f}{\partial x_1}, ..., f\frac{\partial g}{\partial x_n} + g\frac{\partial f}{\partial x_n}\right)\\ =&f\left(\frac{\partial g}{\partial x_1}, ..., \frac{\partial g}{\partial x_n}\right) + g\left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right)\\ =&f\nabla g + g\nabla f.\end{aligned}$$

(*iii*) On a

$$\begin{aligned}\operatorname{div}(fF) =&\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[(fF)_j\right] = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}[fF_j] = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j}F_j + f\frac{\partial F_j}{\partial x_j}\right)\\ =&\underbrace{\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}F_j}_{=\langle\nabla f, F\rangle} + f\underbrace{\sum_{j=1}^{n}\frac{\partial F_j}{\partial x_j}}_{=\operatorname{div} F},\end{aligned}$$

qui est le résultat voulu.

(*iv*) On a

$$\operatorname{rot} F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).$$

Ainsi, (noter que vu que F est C^2, lorsqu'on on prend les dérivées partielles d'ordre 2, l'ordre des variables par rapport à laquelle on dérive n'a pas d'importance)

$$\begin{aligned} \operatorname{rot}\operatorname{rot} F &= \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix} \\ &= \left(\frac{\partial^2 F_2}{\partial x \partial y} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial^2 z} + \frac{\partial^2 F_3}{\partial x \partial z} \right) e_1 \\ &\quad + \left(-\frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial y \partial z} - \frac{\partial^2 F_2}{\partial z^2} \right) e_2 \\ &\quad + \left(\frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y \partial z} \right) e_3. \end{aligned}$$

D'un autre côté, on a

$$\begin{aligned} \frac{\partial}{\partial x} [\operatorname{div} F] &= \frac{\partial}{\partial x} \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right] \\ &= \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} \\ \frac{\partial}{\partial y} [\operatorname{div} F] &= \frac{\partial}{\partial y} \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right] \\ &= \frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y \partial z} \\ \frac{\partial}{\partial z} [\operatorname{div} F] &= \frac{\partial}{\partial z} \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right] \\ &= \frac{\partial^2 F_1}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} + \frac{\partial^2 F_3}{\partial z^2} \end{aligned}$$

Ainsi,

$$\nabla \operatorname{div} F = \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z}, \frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y \partial z}, \frac{\partial^2 F_1}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} + \frac{\partial^2 F_3}{\partial z^2} \right)$$

et donc

$$\begin{aligned} \nabla \operatorname{div} F - \operatorname{rot}\operatorname{rot} F &= \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}, \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2}, \frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2} \right) \\ &= \Delta F. \end{aligned}$$

En réarrangeant les termes, on obtient le résultat voulu.

(v) On a

$$\begin{aligned}\operatorname{rot}(fF) &= \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}[fF_3] - \frac{\partial}{\partial z}[fF_2], \frac{\partial}{\partial z}[fF_1] - \frac{\partial}{\partial x}[fF_3], \frac{\partial}{\partial x}[fF_2] - \frac{\partial}{\partial y}[fF_1]\right) \\ &= f\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\ &\quad + \left(\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2, \frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3, \frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right) \\ &= r \operatorname{rot} F + \left(\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2, \frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3, \frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right).\end{aligned}$$

D'un autre côté,

$$\begin{aligned}\nabla f \wedge F &= \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2, \frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3, \frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right).\end{aligned}$$

En comparant avec ce qu'on a obtenu dans le calcul de $\operatorname{rot}(fF)$, on obtient bien

$$\operatorname{rot}(fF) = \nabla f \wedge F + f \operatorname{rot} F,$$

qui est le résultat voulu.

Exercice 5 (ex 1.4, p. 7, corrigé p. 9)**.**
Soit $f \in C^2(\Omega)$, où

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : \ x, y > 0\right\}.$$

(i) Montrer que, si

$$g\left(r, \theta\right) := f\left(r \cos\theta, r \sin\theta\right) = f(x, y),$$

alors

$$\frac{\partial^2 g(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial g(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g(r,\theta)}{\partial \theta^2} = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = \Delta f(x,y).$$

(ii) Calculer Δf pour

$$f(x, y) := \sqrt{x^2 + y^2} + \left(\arctan \frac{y}{x}\right)^2.$$

Solution :

(i) _Variante 1_ : Calcul direct.

On a

$$
\begin{aligned}
\frac{\partial g}{\partial r}(r,\theta) =& \frac{\partial}{\partial r}\left[f(r\cos\theta, r\sin\theta)\right] \\
=& \frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial r}\left[r\cos\theta\right] + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial r}\left[r\sin\theta\right] \\
=& \frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\cos\theta + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\sin\theta \\
\frac{\partial^2 g}{\partial r^2}(r,\theta) =& \frac{\partial}{\partial r}\left[\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\cos\theta + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\sin\theta\right] \\
=& \frac{\partial}{\partial r}\left[\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\right]\cos\theta + \frac{\partial}{\partial r}\left[\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\right]\sin\theta \\
=& \left(\frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial r}\left[r\cos\theta\right] + \frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial r}\left[r\sin\theta\right]\right)\cos\theta \\
&+ \left(\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial r}\left[r\cos\theta\right] + \frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial r}\left[r\sin\theta\right]\right)\sin\theta \\
=& \frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta)\cos^2\theta + 2\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta)\cos\theta\sin\theta \\
&+ \frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta)\sin^2\theta \\
\frac{\partial g}{\partial \theta}(r,\theta) =& \frac{\partial}{\partial \theta}\left[f(r\cos\theta, r\sin\theta)\right] \\
=& \frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial \theta}\left[r\cos\theta\right] + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial \theta}\left[r\sin\theta\right] \\
=& r\left(-\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\sin\theta + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\cos\theta\right) \\
\frac{\partial^2 g}{\partial \theta^2}(r,\theta) =& \frac{\partial}{\partial \theta}\left[r\left(-\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\sin\theta + \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\cos\theta\right)\right] \\
=& r\left(-\frac{\partial}{\partial \theta}\left[\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\right]\sin\theta - \frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\cos\theta\right. \\
&\quad \left. + \frac{\partial}{\partial \theta}\left[\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\right]\cos\theta - \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\sin\theta\right) \\
=& -r\sin\theta\left(\frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial \theta}\left[r\cos\theta\right] + \frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial \theta}\left[r\sin\theta\right]\right) \\
&- r\cos\theta\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) \\
&+ r\cos\theta\left(\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial \theta}\left[r\cos\theta\right] + \frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta)\frac{\partial}{\partial \theta}\left[r\sin\theta\right]\right) \\
&- r\sin\theta\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta) \\
=& r^2\sin^2\theta\frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta) - 2r^2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}(r\cos\theta, r\sin\theta) \\
&+ r^2\cos^2\theta\frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta) \\
&- r\cos\theta\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) - r\sin\theta\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)
\end{aligned}
$$

Ainsi,

$$\begin{aligned}\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + &\frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}\\ &= \underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{=1}\frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta) + \underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{=1}\frac{\partial^2 f}{\partial y^2}(r\cos\theta, r\sin\theta)\\ &= \Delta f(r\cos\theta, r\sin\theta)\end{aligned}$$

Variante 2 : Avec des feintes de physicien · nes.
Si $x(r,\theta) = r\cos\theta$ et $y(r,\theta) = r\sin\theta$, on a

$$\begin{aligned}\frac{\partial x}{\partial r} &= \cos\theta = \frac{x}{r}\\ \frac{\partial x}{\partial \theta} &= -r\sin\theta = -y\\ \frac{\partial y}{\partial r} &= \sin\theta = \frac{y}{r}\\ \frac{\partial y}{\partial \theta} &= r\cos\theta = x\end{aligned}$$

Ainsi,

$$\begin{aligned}\frac{\partial g}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{1}{r}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)\\ \frac{\partial^2 g}{\partial r^2} &= -\frac{1}{r^2}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)\\ &\quad + \frac{1}{r}\left(\frac{\partial x}{\partial r}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial f}{\partial y} + x\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial x\partial y}\frac{\partial y}{\partial r}\right) + y\left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial r}\right)\right)\\ &= -\frac{1}{r^2}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)\\ &\quad + \frac{1}{r}\left(\frac{x}{r}\frac{\partial f}{\partial x} + \frac{y}{r}\frac{\partial f}{\partial y} + x\left(\frac{\partial^2 f}{\partial x^2}\frac{x}{r} + \frac{\partial^2 f}{\partial x\partial y}\frac{y}{r}\right) + y\left(\frac{\partial^2 f}{\partial x\partial y}\frac{x}{r} + \frac{\partial^2 f}{\partial y^2}\frac{y}{r}\right)\right)\\ &= \frac{1}{r^2}\left(x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right)\\ \frac{\partial g}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}\\ \frac{\partial^2 g}{\partial \theta^2} &= -\frac{\partial y}{\partial \theta}\frac{\partial f}{\partial x} - y\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial x\partial y}\frac{\partial y}{\partial \theta}\right) + \frac{\partial x}{\partial \theta}\frac{\partial f}{\partial y} + x\left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial \theta}\right)\\ &= -x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} - y\left(-y\frac{\partial^2 f}{\partial x^2} + x\frac{\partial^2 f}{\partial x\partial y}\right) + x\left(-y\frac{\partial^2 f}{\partial x\partial y} + x\frac{\partial^2 f}{\partial y^2}\right)\\ &= -x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} + y^2\frac{\partial^2 f}{\partial x^2} - 2xy\frac{\partial^2 f}{\partial x\partial y} + x^2\frac{\partial^2 f}{\partial y^2}\end{aligned}$$

Et donc on conclut

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} = \underbrace{\frac{x^2+y^2}{r^2}}_{=1}\frac{\partial^2 f}{\partial x^2} + \underbrace{\frac{x^2+y^2}{r^2}}_{=1}\frac{\partial^2 f}{\partial y^2} = \Delta f$$

(ii) Posons

$$g(r,\theta) = f(r\cos\theta, r\sin\theta) = r + \left(\arctan\frac{\sin\theta}{\cos\theta}\right)^2 = r + (\arctan\tan)^2 = r + \theta^2$$

Alors,

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} = 0 + \frac{1}{r}\cdot 1 + \frac{1}{r^2}\cdot 2 = \frac{2+r}{r^2} = \Delta f(r\cos\theta, r\sin\theta).$$

Ainsi,

$$\Delta f(x,y) = \frac{2+\sqrt{x^2+y^2}}{x^2+y^2}$$