- **Distribution**: It is a linear, continuous functional on D.

$$T : D \longrightarrow \mathbb{R} \quad \text{s.t.}$$

- T is finite
- T is linear

$$\forall \alpha, \beta \in \mathbb{R}, \quad \forall \varphi, \psi \in D$$

$$T(\alpha\varphi + \beta\psi) = \alpha T(\varphi) + \beta T(\psi)$$

- T is continuous:

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if T changes little, the output changes little.

$$\forall \varphi \in D : \operatorname{supp} \varphi \subseteq [a, b]$$

$$|T(\varphi)| \leq C \sum_{i=0}^{k} \max_{x \in \mathbb{R}} \left| \frac{\partial^i \varphi(x)}{\partial x^i} \right|$$

for each $[a,b] \subset \mathbb{R}$ there exist $C > 0$ and $k \in \mathbb{N}$ s.t. $\nearrow$

Note: $T(\varphi) = \langle T, \varphi \rangle$

7.3 Examples

Example 1: let f be any integrable over $\mathbb{R}$.

$f \in L_1(\mathbb{R})$, i.e., $\int_{-\infty}^{+\infty} |f(x)|\,dx < \infty$.

$$T_f(\varphi) = \langle T_\varphi, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x)\,dx \quad \text{is a distribution.}$$

$\varphi \in D$

$\operatorname{supp} \varphi \subseteq [a,b]$

$$= \int_a^b f(x)\varphi(x)\,dx$$

- $\langle f, \varphi \rangle$ is finite.

- $\langle f, \varphi \rangle$ is linear.

$$\langle f, \alpha\varphi + \beta\psi \rangle = \int_{\mathbb{R}} (\alpha\varphi + \beta\psi) f \, dx$$

$$= \alpha \int_{\mathbb{R}} \varphi(x) f(x) \, dx + \beta \int_{\mathbb{R}} \psi(x) f(x) \, dx.$$

$$= \alpha \langle f, \varphi \rangle + \beta \langle f, \psi \rangle.$$

Note: by abuse of notation $\langle f, \varphi \rangle = \langle T_f, \varphi \rangle$.

- $\langle T_f, \varphi \rangle$ is continuous.

$$|\langle T_f, \varphi \rangle| = \left| \int_{-\infty}^{+\infty} f(x)\varphi(x) \, dx \right| \leq \underbrace{\int_{-\infty}^{+\infty} |f(x)| \, dx}_{C} \max_{x \in \mathbb{R}} |\varphi(x)|$$

$k=0$

$$\leq \int_{-\infty}^{+\infty} |f(x)\varphi(x)| \, dx \leq \int_{-\infty}^{+\infty} |f(x)| \, |\varphi(x)| \, dx \leq \int_{-\infty}^{+\infty} |f(x)| \, dx \max_{x \in \mathbb{R}} |\varphi(x)|$$

Distribution that derive from integrable functions are called <u>regular distributions</u>.

<u>Example 2:</u> Dirac - Delta

$$\delta_0 : \mathcal{D} \to \mathbb{R} \qquad \text{is distribution.}$$
$$\varphi \mapsto \varphi(0)$$

- it is finite $|\varphi(0)| < \infty$
- it is linear.

$$\delta_0(\alpha\varphi + \beta\psi) = \alpha\,\varphi(0) + \beta\,\psi(0) = \alpha\,\delta_0(\varphi) + \beta\,\delta_0(\psi)$$

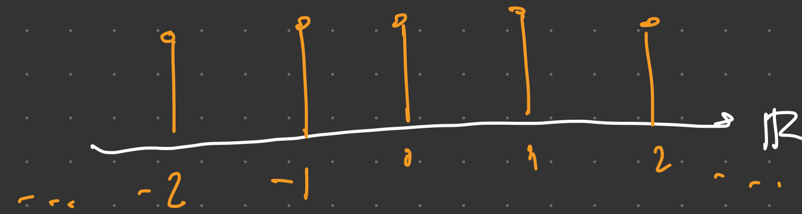
- it is continuous. $\forall \varphi \in \mathcal{D}$

$C=1$ $\quad$ $K=0$.

$$|\delta_0(\varphi)| = |\varphi(0)| \leq \max_{x\in\mathbb{R}} |\varphi(x)|$$

Example 3 : Dirac comb.

$$\Delta_1(\varphi) = \sum_{n\in\mathbb{Z}} \varphi(n)$$



- finite

supp $\varphi \subseteq [a,b]$ is bounded.

$$\Delta_1(\varphi) = \sum_{n\in\mathbb{Z}} \varphi(n) = \sum_{\substack{n\in\mathbb{Z}\\ n\in[a,b]}} \varphi(n) < \infty$$

there is a finite number of $n \in [a,b]$

- linear

$$\Delta_1(\alpha\varphi + \beta\psi) = \sum_{n\in\mathbb{Z}} (\alpha\varphi(n) + \beta\psi(n))$$

$$= \alpha \sum_{n\in\mathbb{Z}} \varphi(n) + \beta \sum_{n\in\mathbb{Z}} \psi(n) = \alpha\Delta_1(\varphi) + \beta\Delta_1(\psi)$$

• continuous.

$$|\Delta_1(\varphi)| = \Big|\sum_{n\in\mathbb{Z}} \varphi(n)\Big| = \Big|\sum_{\substack{n\in\mathbb{Z}\\ n\in[a,b]}} \varphi(n)\Big|$$

$$\leq \underbrace{\sum_{\substack{n\in\mathbb{Z}\\ n\in[a,b]}}}_{\text{M term} < \infty} |\varphi(n)| \leq \underbrace{M}_{C} \max_{x\in\mathbb{R}} |\varphi(x)| \quad k=0.$$

<u>Example 4:</u>

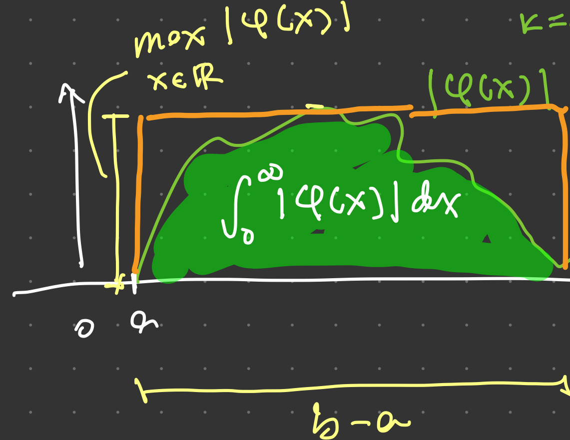
$$\langle T, \varphi\rangle = \int_0^\infty \varphi(x)\,dx$$

• finite $\quad |\langle T,\varphi\rangle| = \left|\int_0^\infty \varphi(x)\,dx\right| \leq \int_0^\infty |\varphi(x)|\,dx$

$$\leq \underbrace{(b-a)}_{c} \max_{x\in\mathbb{R}} |\varphi(x)|$$ ← bounded and continuous.

$k=0$.

Note



$$\leq \boxed{(b-a)\max_{x\in\mathbb{R}}|\varphi(x)|}$$

• linear:

$$\langle T_1, \alpha\varphi+\beta\psi\rangle = \alpha\int_0^\infty \varphi(x)\,dx + \beta\int_0^\infty \psi(x)\,dx = \alpha\langle T_1,\varphi\rangle + \beta\langle T_1,\psi\rangle$$

7.4 Derivatives of distributions

Let $f \in C^\infty(\mathbb{R})$ and $\varphi \in D$ with $\operatorname{supp}\varphi \subseteq [a,b]$. Then

$$\int_{-\infty}^{+\infty} f'(x)\varphi(x)\,dx = \int_a^b f'(x)\varphi(x)\,dx + \underbrace{\int_{-\infty}^{a} f'(x)\varphi(x)\,dx + \int_b^{+\infty} f'(x)\varphi(x)\,dx}_{=0}$$

integrate by parts

$$= f(b)\underbrace{\varphi(b)}_{=0} - f(a)\underbrace{\varphi(a)}_{=0} - \int_a^b f(x)\varphi'(x)\,dx$$

$$= -\int_a^b f(x)\varphi'(x)\,dx = -\int_{-\infty}^{+\infty} f(x)\varphi'(x)\,dx. = \int_{-\infty}^{+\infty} f'(x)\varphi(x)\,dx$$

Inspired by this idea, for

$$\langle T_{f'}, \varphi \rangle = \int_{-\infty}^{+\infty} f'(x)\varphi(x)\,dx = -\int_{-\infty}^{+\infty} f(x)\varphi'(x)\,dx = -\langle T_f, \varphi' \rangle$$

this is the distributional derivative

Example 1: let $f \in C^1(\mathbb{R})$, then T_f is a distribution and its distributional derivative is just its regular derivative

Example 2: let $T : D \to \mathbb{R}$

$$\varphi \mapsto \int_0^\infty \varphi(x)\,dx$$

fundamental theorem of calculus

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle = -\int_0^\infty \varphi'(x)\,dx = \varphi(0) - \underbrace{\lim_{x\to+\infty} \varphi(x)}_{=0}$$

$$= \varphi(0) = \delta_0(\varphi)$$

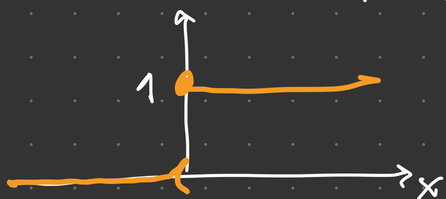
$$= \langle \delta_0, \varphi \rangle$$

Note: $\lim_{x\to\pm\infty} \varphi(x) = 0$ because $\operatorname{supp}\varphi$ is bounded.

$$\langle T, \varphi \rangle = \int_0^{\infty} \varphi(x)\,dx = \int_{-\infty}^{+\infty} H(x)\varphi(x)\,dx$$

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

$$\langle T', \varphi \rangle = \langle \delta_0, \varphi \rangle.$$



$$\langle T_H, \varphi \rangle = \int_{-\infty}^{+\infty} H(x)\varphi(x)\,dx$$

$$\langle T_H', \varphi \rangle = \langle \delta_0, \varphi \rangle.$$

<u>Example 3:</u>

$$T : D \longrightarrow \mathbb{R}$$
$$\varphi \longmapsto \int_0^1 \varphi(x)\,dx.$$

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle = -\int_0^1 \varphi'(x)\,dx = \varphi(0) - \varphi(1)$$

$$= \langle \delta_0, \varphi \rangle - \langle \delta_1, \varphi \rangle$$

corrected after lecture

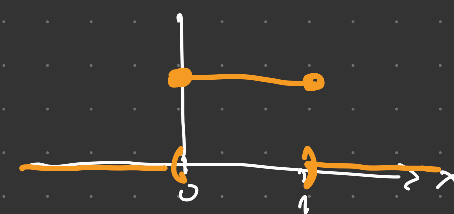
Another perspective

$$T : \mathcal{D} \longrightarrow \mathbb{R}$$

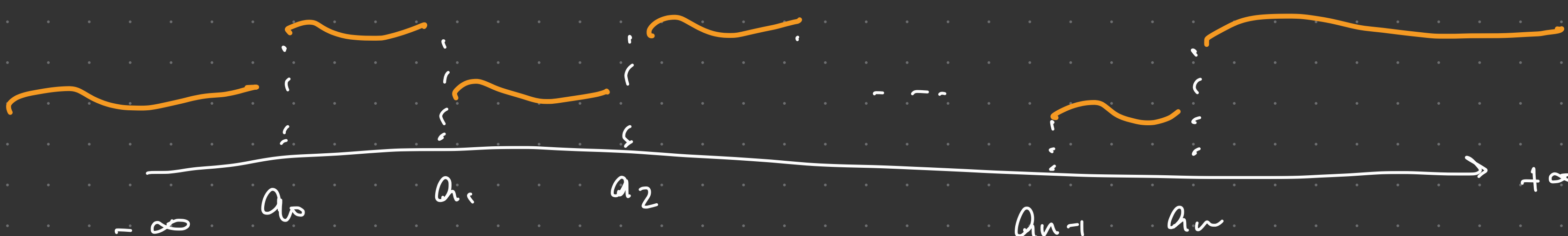
$$\varphi \longmapsto \int_{-\infty}^{+\infty} g(x)\,\varphi(x)\,dx.$$

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{else.} \end{cases}$$

$$= \int_0^1 \varphi(x)\,dx.$$



Example 4: Let $f:\mathbb{R}\to\mathbb{R}$ is piecewise defined and differentiable over the segments $(-\infty, a_0)$, (a_0, a_1), (a_1, a_2) $\dots$ (a_{n-1}, a_n), (a_n, ∞)



$$\langle T_f, \varphi\rangle = \int_{-\infty}^{+\infty} f(x)\varphi(x)\,dx.$$

$$\langle T_{f'}, \varphi\rangle = -\langle T_f, \varphi'\rangle = -\int_{-\infty}^{+\infty} f(x)\varphi'(x)\,dx$$

$$= -\int_{-\infty}^{a_0} f(x)\varphi'(x) - \sum_{i=1}^{n}\int_{a_{i-1}}^{a_i} f(x)\varphi'(x)\,dx - \int_{a_n}^{+\infty} f(x)\varphi'(x)\,dx.$$

$$= -f(a_0-0)\varphi(a_0) + \int_{-\infty}^{a_0} f'(x)\varphi(x)\,dx + \lim_{x\to-\infty} f(x)\varphi(x)$$

by parts ↗

$$f(a+0) = \lim_{x\to a^+} f(x)$$

$$f(a-0) = \lim_{x\to a^-} f(x)$$

$$-\sum_{i=1}^{n}\left[f(a_i-0)\varphi(a_i) - f(a_{i-1}+0)\varphi(a_{i-1}) - \int_{a_{i-1}}^{a_i} f'(x)\varphi(x)\,dx \right]$$

in green, corrections after lecture

$$+ f(a_n+0)\varphi(a_n) + \int_{a_n}^{\infty} f'(x)\varphi(x)\,dx.$$

$$= \int_{-\infty}^{a_0} f'(x)\varphi(x)\,dx + \sum_{i=1}^{n}\int_{a_{i-1}}^{a_i} f'(x)\varphi(x)\,dx + \int_{a_n}^{+\infty} f'(x)\varphi(x)\,dx$$

$$+ \sum_{i=0}^{n} \varphi(a_i)\Big[\underbrace{f(a_i+0) - f(a_i-0)}_{\text{jump of } f \text{ at } a_i}\Big]$$

Remark: if f is piecewise continuous: all the jumps are zero.

$$\langle f', \varphi \rangle = \int_{-\infty}^{a_0} f'(x)\varphi(x)\,dx + \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} f'(x)\varphi(x)\,dx + \int_{a_n}^{\infty} f'(x)\varphi(x)\,dx$$

$$= \int_{-\infty}^{+\infty} f'(x)\varphi(x)\,dx.$$

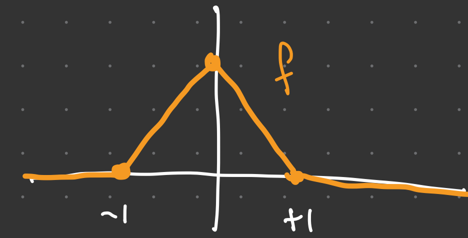
$f \notin C^1(\mathbb{R})$
only piecewise.

This function f' is called <u>weak derivative</u> of f.

Classical derivatives
Analyse I, II ... $\subseteq$ weak derivatives (piecewise derivatives of continuous differentiable piecewise fncs) $\subseteq$ distributional derivatives.

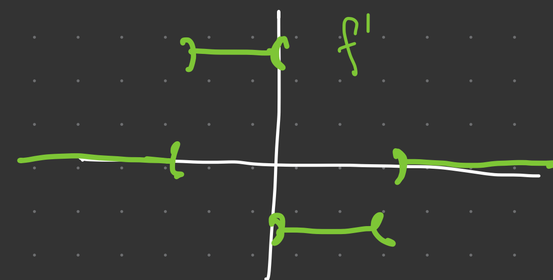
Example 5

$$f(x) = \begin{cases} 0 & x < -1 \\ x+1 & -1 \leq x \leq 0 \\ -x+1 & 0 \leq x \leq 1 \\ 0 & 1 < x \end{cases}$$

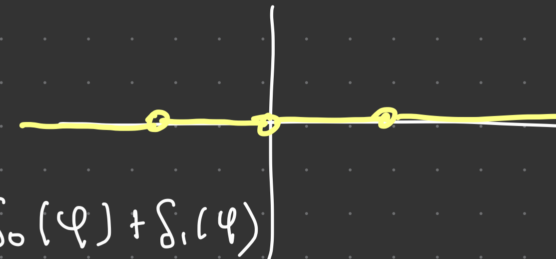


$$\langle T_{f'}, \varphi \rangle =$$

$$f'(x) = \begin{cases} 0 & x < -1 \\ 1 & -1 \leq x \leq 0 \\ -1 & 0 \leq x \leq 1 \\ 0 & 1 < x \end{cases}$$



$$\langle T_{f''}, \varphi \rangle = - \langle T_{f'}, \varphi' \rangle$$



$$= - \int_{-\infty}^{+\infty} f'(x)\varphi'(x) = \dots = \delta_{-1}(\varphi) - 2\delta_0(\varphi) + \delta_1(\varphi)$$