- Imposing initial condition $u(x,0) = f(x)$

$$\hat{u}(\alpha,0) = \hat{f}(\alpha) = \underbrace{\exp(-a^2\alpha^2 0)}_{=1} c(\alpha) = \hat{f}(\alpha)$$

$$c(\alpha) = \hat{f}(\alpha)$$

$$\hat{u}(\alpha,t) = \exp(-a^2\alpha^2 t)\,\hat{f}(\alpha).$$

Apply inverse Fourier transf.

Note:
you can compute $\hat{f}(\alpha)$ using table
$\hat{f}(\alpha) = \frac{1}{\sqrt{2}} e^{-\alpha^2/4}$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2}} \exp(-a^2\alpha^2 t) \exp(-\alpha^2/4)\, e^{+i\alpha x}\, d\alpha.$$

9/12/2025

$$u(x,t) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\alpha^2\left(\frac{1}{4} + a^2 t\right)\right) e^{+i\alpha x}\, d\alpha$$

α →

$$= \frac{1}{\sqrt{2}} \mathcal{F}\left(\exp\left(-\alpha^2\overbrace{\left(\frac{1}{4} + a^2 t\right)}^{\omega^2}\right)\right)(-x,t)$$

Table of Fourier transform.

8: $f(y) = e^{-\omega^2 y^2}$ $\quad \omega \neq 0 \quad \Big| \quad \hat{f}(\alpha) = \dfrac{1}{\sqrt{2}\,|\omega|} e^{-\frac{\alpha^2}{4\omega^2}}$

$$u(x,t) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}\sqrt{\frac{1}{4}+a^2t}} \exp\left(-\frac{(-x)^2}{1+4a^2t}\right)$$

$$u(x,t) = \frac{9}{\sqrt{1+4a^2t}} e^{-\frac{x^2}{1+4a^2t}} \quad \forall\ t \in \mathbb{R}^+.$$

$$\rightarrow u(-x,t) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\Big(-\alpha^2 \underbrace{\left(\tfrac{1}{4}+a^2t\right)}_{\omega^2}\Big) e^{-i\alpha x}\, d\alpha$$

$\rightarrow$

6.3.3 Wave equation for a finite length bar

Let be $c, L \in \mathbb{R}^+$, $f, g : [0, L] \to \mathbb{R}$, $f, g \in C^3([0,L])$

s.t. $f(0) = f(L) = 0$, and $g(0) = g(L) = 0$.

find the solution of

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \qquad \forall\, x \in]0, L[\,,\ t > 0.$$

$$u(0,t) = u(L,0) = 0 \qquad t > 0$$

initial conditions
$$\begin{cases} u(x,0) = f(x) & x \in]0, L[\\ \dfrac{\partial u}{\partial t}(x,0) = g(x) & x \in]0, L[\end{cases}$$

Discuss this for the case $f = 0$ and $g(x) = \sin\left(\frac{\pi}{L} x\right) - \sin\left(\frac{2\pi}{L} x\right)$

let's focus on

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \qquad \forall\, x \in]0,L[\,,\ t>0.$$

$$u(0,t) = u(L,0) = 0 \qquad t>0$$

We look for solutions like $u(x,t) = v(x)\,w(t)$

$$v(x)\,w''(t) = c^2 v''(x)\,w(t) \qquad \forall\, x \in]0,L[\,,\ t>0$$

$$v(0)\,w(t) = v(L)\,w(t) = 0 \qquad t>0$$

$$\longrightarrow v(0) = v(L) = 0.$$

My new problem:

$$\frac{v''(x)}{v(x)} = -\lambda = \frac{1}{c^2}\frac{w''(t)}{w(t)} \quad \text{and} \quad v(0)=v(L)=0$$

1st $\quad v''(x) + \lambda v(x) = 0 \qquad x \in]0, L[$

$v(0) = v(L) = 0$

} Sturm-Liouville.

2nd $\quad w''(t) + c^2 \lambda w(t) = 0 \qquad t > 0$

1st $\rightarrow \quad \lambda = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, \ldots$

$$v_n(x) = \sin\left(\frac{n\pi}{L} x\right).$$

2nd $\rightarrow \quad w''_n(t) + \left(c \frac{n\pi}{L}\right)^2 w(t) = 0$

$$w_n(t) = a_n \cos\left(\frac{\pi n c}{L} t\right) + b_n \sin\left(\frac{\pi n c}{L} t\right)$$

The solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right)\right] \sin\left(\frac{n\pi}{L} x\right)$$

we impose initial conditions $\begin{cases} u(x,0) = f(x) \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$

- $u(x,0) = f(x) = \sum_{n=1}^{\infty} \left[a_n \cdot 1 + b_n \cdot 0 \right] \sin\left(\frac{n\pi}{L} x\right)$

$$= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L} x\right)$$

This is the Fourier series in sines of $f(x)$.

$$a_n = \frac{2}{L} \int_0^L f(y) \sin\left(\frac{n\pi}{L} y\right) dy$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \left[\frac{n\pi c}{L} a_n \cdot 0 + \frac{n\pi c}{L} b_n \cdot 1 \right] \sin\left(\frac{n\pi}{L} x\right) = g(x)$$

$$= \sum_{n=1}^{\infty} \left[\frac{n\pi c}{L} b_n \sin\left(\frac{n\pi}{L} x\right) \right]$$

$$\frac{n\pi c}{L} b_n = \frac{2}{L}\int_0^L g(y)\, \sin\left(\frac{n\pi}{L} y\right) dy$$

$$b_n = \frac{2}{n\pi c}\int_0^L g(y)\, \sin\left(\frac{n\pi}{L} y\right) dy.$$

- $f(x)=0$ and $g(x) = \sin\left(\frac{\pi}{L}x\right) - \sin\left(\frac{2\pi}{L}x\right)$

$f(x)=0 \longrightarrow a_n = 0 \ \forall n,$

$g(x) = \sin\left(\frac{\pi}{L}x\right) - \sin\left(\frac{2\pi}{L}x\right)$

$\longrightarrow \ b_1 \frac{\pi c}{L} = 1 \longrightarrow b_1 = \frac{L}{\pi c}$

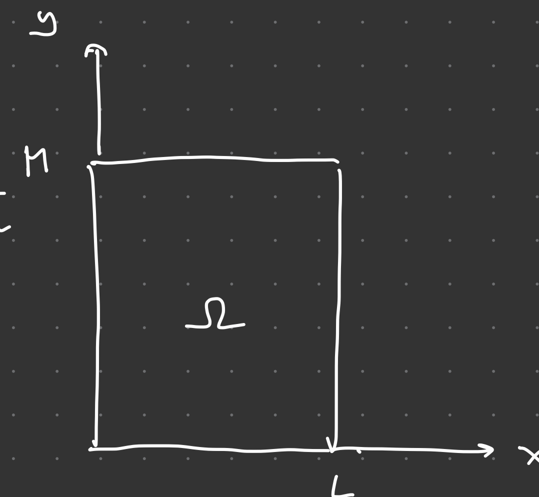
$\longrightarrow \ b_2 \frac{2\pi c}{L} = -1 \longrightarrow b_2 = -\frac{L}{2\pi c}$

Then:

$$u(x,t) = \frac{L}{\pi c} \sin\left(\frac{\pi c}{L} t\right) \sin\left(\frac{\pi}{L} x\right) - \frac{L}{2\pi c} \sin\left(\frac{2\pi c}{L} t\right) \sin\left(\frac{2\pi}{L} x\right).$$

6.3.4 Laplace equation in a rectangle

$$\Omega =]0,L[\times]0,M[$$



$$\Delta u(x,y) = 0$$

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0$$

Poisson problem $\Delta u + f = 0$

let be, $L, M \in \mathbb{R}^+$, $\alpha, \beta \in [0, L] \to \mathbb{R}$, $\gamma, \delta : [0, M] \to \mathbb{R}$.

s.t. $\alpha, \beta \in C^1([0, L])$, $\gamma, \delta \in C^1([0, M])$

and $\alpha(0) = \alpha(L) = \beta(0) = \beta(L) = 0$

$\gamma(0) = \gamma(M) = \delta(0) = \delta(L) = 0.$

$$
\begin{cases}
\Delta u(x,y) = \dfrac{\partial^2 u}{\partial x^2}(x,y) + \dfrac{\partial^2 u}{\partial y^2}(x,y) = 0 & \forall (x,y) \in \Omega \\
u(x,0) = \alpha(x) & x \in]0, L[\\
u(x,M) = \beta(x) & \\
u(0,y) = \gamma(y) & y \in]0, M[\\
u(L,y) = \delta(y) &
\end{cases} \qquad (0)
$$

Particularize for $\alpha(x) = 4\sin\left(\frac{\pi x}{L}\right)$ and $\beta(x) = -\sin\left(\frac{2\pi x}{L}\right)$

$\gamma(y) = \delta(y) = 0.$

$$u(x,y) = v(x,y) + w(x,y)$$

s.t. $v(x,y)$ is the solution

$$\begin{cases} \Delta v(x,y) = 0 & \forall (x,y) \text{ in } \Omega \\ v(x,0) = 0 = v(x,M) & x \in]0,L[\\ v(0,y) = \gamma(y),\ v(L,y) = \delta(y) & y \in]0,M[\end{cases} \quad (1)$$

and s.t. $w(x,y)$ is the solution

$$\begin{cases} \Delta w(x,y) = 0 & \forall (x,y) \in \Omega \\ w(x,0) = \alpha(x),\ w(x,M) = \beta(x) & x \in]0,L[\\ w(0,y) = w(L,y) = 0 & y \in]0,M[\end{cases} \quad (2)$$

If v is solution of (1) and w is solution (2), $v+w$ is solution of (0)

(1) and (2) are analogous, so I just study (2).

I assume $w(x,y) = f(x)g(y)$ and introduce in (2)

$$\begin{cases} f''(x)g(y) + f(x)g''(y) = 0 & \forall (x,y) \text{ in } \Omega \\ f(x)g(0) = \alpha(x) \quad , \quad f(x)g(M) = \beta(x) \\ f(0)g(y) = 0 \quad , \quad f(L)g(y) = 0 \quad , \quad f(0)=0, \; f(L)=0. \end{cases}$$

$$\begin{cases} \dfrac{f''(x)}{f(x)} = -\lambda = -\dfrac{g''(y)}{g(y)} & \forall (x,y) \in \Omega \\ f(0) = f(L) = 0. \end{cases}$$

$$\begin{cases} f''(x) + \lambda f(x) = 0 \\ f(0) = f(L) = 0 \end{cases}$$

Sturm-Liouville

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \qquad f_n(x) = \sin\left(\frac{n\pi}{L} x\right)$$

$$\begin{cases} g''(y) - \left(\frac{n\pi}{L}\right)^2 g(y) = 0 \\ \text{no bcs (so far)} \end{cases}$$

$$g_n(y) = a_n \cosh\left(\frac{n\pi}{L} y\right) + b_n \sinh\left(\frac{n\pi}{L} y\right)$$

$$w(x,y) = \sum_{n=1}^{\infty} \left[a_n \cosh\left(\frac{n\pi}{L} y\right) + b \sinh\left(\frac{n\pi}{L} y\right)\right] \sin\left(\frac{n\pi}{L} x\right)$$

Fn introduce $f(x) g(0) = \alpha(x)$, $f(x) g(M) = \beta(x)$

$$\bullet\ w(x,0) = \sum_{n=1}^{\infty}\left[a_n \cdot 1 + b_n \cdot 0\right] \sin\left(\frac{n\pi}{L}x\right) = \alpha(x)$$

$$\alpha(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) \rightarrow \text{This is the } F_S \text{ of } \alpha(x).$$

corrected after lecture

$$a_n = \frac{2}{L}\int_0^L \alpha(t)\sin\left(\frac{n\pi}{L}t\right)dt$$

$$\bullet\ w(x,M) = \sum_{n=1}^{\infty}\overbrace{\left[a_n \cosh\left(\frac{n\pi}{L}M\right) + b_n \sinh\left(\frac{n\pi}{L}M\right)\right]}^{c_n} \sin\left(\frac{n\pi}{L}x\right)$$

$$= \beta(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)$$

$$c_n = a_n \cosh\left(\frac{n\pi}{L}M\right) + b_n \sinh\left(\frac{n\pi}{L}M\right),$$ knowing a_n, knowing c_n $\rightarrow b_n$

$$c_n = \frac{2}{L}\int_0^L \beta(t)\sin\left(\frac{n\pi}{L}t\right)dt.$$

Once computed a_n and b_n, my solution is:

$$w(x,y) = \sum_{n=1}^{\infty} \left[a_n \cosh\left(\frac{n\pi}{L} y\right) + b \sinh\left(\frac{n\pi}{L} y\right)\right] \sin\left(\frac{n\pi}{L} x\right)$$

Doing the same for problem (1) we get

$$v(x,y) = \sum_{n=1}^{\infty} \left[d_n \cosh\left(\frac{n\pi}{M} x\right) + e_n \sinh\left(\frac{n\pi}{M} x\right)\right] \sin\left(\frac{n\pi}{M} y\right)$$

$$d_n = \frac{2}{M} \int_0^M \gamma(t) \sin\left(\frac{n\pi}{M} t\right) dt$$

corrected after lecture

$$f_n = \frac{2}{M} \int_0^M \delta(t) \sin\left(\frac{n\pi}{M} t\right) dt$$

$$e_n \sinh\left(\frac{n\pi}{M} L\right) = f_n - d_n \cosh\left(\frac{n\pi}{M} L\right).$$

For the particular case:

$$\alpha(x) = 4\sin\left(\frac{\pi x}{L}\right), \quad \beta(x) = -\sin\left(\frac{2\pi x}{L}\right)$$

$$\gamma = \delta = 0 \longrightarrow d_n = 0, \ f_n = 0, \ e_n = 0 \quad \forall n.$$

$$a_1 = 4, \quad a_n = 0 \quad \forall n > 1$$

$$c_2 = -1, \quad c_n = 0 \quad \forall n \text{ s.t } n \neq 2$$

$$b_1 = -4\coth\left(\frac{\pi}{L}M\right), \quad b_2 = -1/\sinh\left(\frac{2\pi}{L}M\right)$$

$$b_n = 0 \quad \forall n > 2.$$

The final solution:

$$4\left[\cosh\left(\frac{\pi}{L}y\right) - \coth\left(\frac{\pi}{L}M\right)\sinh\left(\frac{\pi}{L}y\right)\right]\sin\left(\frac{\pi}{L}x\right)$$

$$-\frac{1}{\sinh\left(\frac{2\pi}{L}M\right)}\sinh\left(\frac{2\pi}{L}y\right)\sin\left(\frac{2\pi}{L}x\right) = u(x,y).$$

6.3.5 Laplace equation in a disk

Find u: $\Delta u(x,y) = 0$ in Ω, $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < R^2\}$.

$\varphi : \partial\Omega \to \mathbb{R}$, s.t. $\varphi \in C^1(\partial\Omega)$

Polar coordinates:

$x = r\cos\theta$, $y = r\sin\theta$

$$u(x,y) = u(r\cos\theta, r\sin\theta) = V(r,\theta)$$

$$\Delta V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

+ bcs.