2/12/2025.

For bc

- $u(0) = 0 = \frac{u_1}{\sqrt{-\lambda}} \underbrace{\sinh(0\sqrt{-\lambda})}_{=0}$ ✓

- $u(L) = 0 = \frac{u_1}{\sqrt{-\lambda}} \underbrace{\sinh(L\sqrt{-\lambda})}_{\neq 0} \Longrightarrow u_1 = 0$

$u(x) = 0$ trivial solution!

- Case 3: $\lambda > 0$

$$u''(x) + \lambda u(x) = 0 \quad \text{in} \quad x \in]0, L[$$

$$u(0) = u(L) = 0$$

$$u(x) = \frac{u_1}{\sqrt{\lambda}} \sin(x\sqrt{\lambda}) \quad , \quad u'(x) = u_1 \cos(x\sqrt{\lambda})$$

$$u''(x) = -u_1 \sqrt{\lambda} \sin(x\sqrt{\lambda})$$

$$u''(x) + \lambda u(x) = -u_1 \sqrt{\lambda} \sin(x\sqrt{\lambda}) + \lambda \frac{u_1}{\sqrt{\lambda}} \sin(x\sqrt{\lambda}) = 0 \checkmark$$

- $$u(0) = \frac{u_1}{\sqrt{\lambda}} \underbrace{\sin(0\sqrt{\lambda})}_{=0} = 0 \checkmark$$

- $$u(L) = \frac{u_1}{\sqrt{\lambda}} \underbrace{\sin(L\sqrt{\lambda})}_{=0} = 0 \Rightarrow L\sqrt{\lambda} = n\pi \; , \; n \in \mathbb{Z}$$

$(n = -\infty, \dots, 0, 1, \dots \infty)$

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

$$u(x) = \alpha_n \sin\left(\frac{n\pi}{L} x\right) \quad \text{where } \alpha_n \in \mathbb{R} \quad \forall n \in \mathbb{Z}$$

Example: changing boundary conditions:

$$u''(x) + \lambda u(x) = 0 \quad \forall x \in]0, L[\qquad \lambda \in \mathbb{R}$$

$$u'(0) = 0 = u'(L)$$

- Case 1: $\lambda = 0$:

$$u''(x) = 0 \longrightarrow u(x) = u_1 x + u_0 \ , \ u'(x) = u_1$$

$\rightarrow$ Impose $u'(0) = u'(L) = 0 \longrightarrow u_1 = 0$

$u_0 \in \mathbb{R}$

$$u(x) = u_0$$

- Case $\lambda < 0$:

$$u(x) = u_0 \cosh(x\sqrt{-\lambda}) \ , \ u'(x) = u_0 \sqrt{-\lambda} \sinh(x\sqrt{-\lambda})$$

$$u''(x) = -u_0 \lambda \cosh(x\sqrt{-\lambda})$$

$$u''(x) + \lambda u(x) = \underbrace{(-u_0 \lambda + \lambda u_0)}_{=0} \cosh(x\sqrt{-\lambda}) = 0 \checkmark$$

→ bcs

$$u'(0) = u_0 \sqrt{-\lambda}\, \underbrace{\sinh(0\sqrt{-\lambda})}_{=0} = 0 \checkmark$$

$$u'(L) = u_0 \sqrt{-\lambda}\, \underbrace{\sinh(L\sqrt{-\lambda})}_{\neq 0} = 0 \rightarrow u_0 = 0$$

trivial solution!

- case 3: $\lambda > 0$

$$u(x) = u_0 \cos(x\sqrt{\lambda}), \quad u'(x) = -u_0\sqrt{\lambda}\sin(x\sqrt{\lambda})$$

$$u''(x) = -u_0 \lambda \cos(x\sqrt{\lambda})$$

$$u''(x) + \lambda u(x) = 0 \quad \checkmark$$

$\rightarrow$ bcs:

$$u'(0) = -u_0\sqrt{\lambda}\,\underbrace{\sin(0\sqrt{\lambda})}_{=0} = 0 \checkmark$$

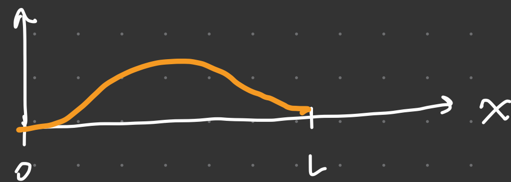
$$u'(L) = -u_0\sqrt{\lambda}\,\underbrace{\sin(L\sqrt{\lambda})}_{=0} = 0 \longrightarrow L\sqrt{\lambda} = n\pi, \; n \in \mathbb{Z}$$

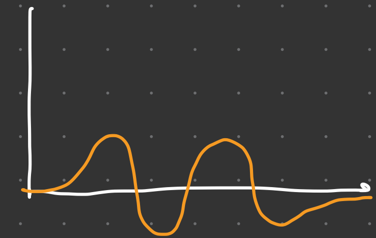
$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

$$u(x) = \beta_n \cos\left(\frac{n\pi}{L} x\right) \text{ where } \beta_n \in \mathbb{R}.$$

$$u''(x) + \lambda u(x) = 0$$

$$u'(0) = 0 = u'(L)$$





6.2.3 Application of Fourier series

study differential equations where the additional data is of the form $u(0) = u(T)$, $u(x) = u(x+T)$, $T \in \mathbb{R}^+$

$\Longrightarrow$ I.e., u is T-periodic.

Example : Heat diffusion of very thin plate

$$\frac{\partial^2 u}{\partial \theta^2} = R^2 f(R\cos\theta, R\sin\theta)$$

$$u(0) = u(2\pi)$$

$$u'(0) = u'(2\pi)$$

u is the temperature at every point.

$(x,y) = R(\cos\theta, \sin\theta)$

Example 1:

let be $f : \mathbb{R} \to \mathbb{R}$, $f \in C^1(\mathbb{R})$ and 2π - periodic

let $m, K \in \mathbb{R}$, s.t. $m \neq 0$ $(m \in \mathbb{R}^*)$. Find a solution $u = u(x)$ of the problem:

$$m u''(x) + K u(x) = f(x) \quad \forall x \in \left]0, 2\pi\right[$$

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi).$$

Particularize the solution for $f(x) = \cos x$.

$$u(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left[\alpha_n \cos nx + \beta_n \sin nx \right] = Ff(x) \quad \text{(Dirichlet)}$$

α_n, β_n are "known"; a_n, b_n are my unknown.

$$u'(x) = \sum_{n=1}^{\infty} \left(-a_n n \sin nx + b_n n \cos nx \right)$$

$$u''(x) = - \sum_{n=1}^{\infty} \left(a_n n^2 \cos nx + b_n n^2 \sin nx \right)$$

$$m u''(x) + K u(x) = f(x)$$

$$m (F u(x))'' + K F u(x) = F f(x)$$

$$\frac{K a_0}{2} + \sum_{n=1}^{\infty} \left[(K - mn^2) a_n \cos(nx) + (K - mn^2) b_n \sin(nx) \right]$$

$$= \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left[\alpha_n \cos(nx) + \beta_n \sin(nx) \right]$$

- $\frac{K a_0}{2} = \frac{\alpha_0}{2} \longrightarrow a_0 = \frac{\alpha_0}{K}$

- $(K - mn^2) a_n = \alpha_n \longrightarrow a_n = \frac{\alpha_n}{K - mn^2}$

- $(K - mn^2) b_n = \beta_n \longrightarrow b_n = \frac{\beta_n}{K - mn^2}$

$$f(x) = \cos x \longrightarrow \begin{cases} \alpha_0 = 0 \\ \alpha_1 = 1, \quad \alpha_n = 0 \quad \forall\, n \geq 2 \\ \beta_n = 0 \quad \forall\, n \geq 1. \end{cases}$$

$$a_0 = 0, \quad a_1 = \frac{1}{K - m}, \quad a_n = 0 \quad \forall\, n \geq 2, \quad b_n = 0 \quad \forall\, n \geq 1.$$

The solution is $u(x) = \frac{1}{K-m} \cos x.$

• Example 2:

let be $K \neq \pm 1$ and $f: \mathbb{R} \to \mathbb{R}$ 2π-periodic and piecewise defined in $[0, 2\pi]$. Find $u: [0, 2\pi]$ s.t.

$$u(x) + K u(x-\pi) = f(x) \quad \forall x \in]0, 2\pi[$$

$$u(0) = u(2\pi)$$

$$u(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

(fulfills $u(0) = u(2\pi)$)

$$u(x-\pi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n(x-\pi)) + b_n \sin(n(x-\pi)) \right)$$

$$\cos(n(x-\pi)) = \cos nx \cos n\pi + \sin nx \cancel{\sin n\pi}^{\,0} = (-1)^n \cos nx$$

$$\sin(n(x-\pi)) = \sin nx \cos n\pi - \cos nx \underbrace{\cancel{\sin n\pi}}_{0} = (-1)^n \sin nx$$

$$u(x-\pi) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left((-1)^n a_n \cos nx + (-1)^n b_n \sin nx \right)$$

$$\frac{1+K}{2} a_0 + \sum_{n=1}^{\infty}\left[(1+K(-1)^n) a_n \cos nx + (1+K(-1)^n) b_n \sin nx \right]$$

$$= F f(x) = f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}\left(\alpha_n \cos nx + \beta_n \sin nx \right)$$

Dirichlet
where $f(x)$ is continuous.

$$a_0 = \frac{1}{1+K}\alpha_0, \quad a_n = \frac{1}{1+K(-1)^n}\alpha_n, \quad b_n = \frac{1}{1+K(-1)^n}\beta_n$$

If, for instance, $f(x) = \cos x + 3\sin 2x + 4\cos 5x$

$$\alpha_1 = 1, \ \beta_2 = 3, \ \alpha_5 = 4$$

$$u(x) = \frac{1}{1-K}\cos x + \frac{4}{1-K}\cos 5x + \frac{3}{1+K}\sin 2x.$$

6.2.4 Application of Fourier transform to ODEs

Recall:

- $\mathcal{F}(af+bg)(\alpha) = a\hat{f}(\alpha) + b\hat{g}(\alpha)$
- $\mathcal{F}(f')(\alpha) = i\alpha\hat{f}(\alpha)$, $\quad \mathcal{F}(f^{(k)})(\alpha) = (i\alpha)^k\hat{f}(\alpha)$
- $\mathcal{F}(f * g)(\alpha) = \sqrt{2\pi}\,\hat{f}(\alpha)\hat{g}(\alpha)$ $\qquad$ • shift ...

Example: Find the solution $u(x)$ of:

- $$9u(x) + \int_{-\infty}^{+\infty} 8u(t)\, e^{-|x-t|}\, dt = e^{-|x|}$$ (integral equation)

...

- $$u''(x) - u(x) = e^{-|x|}$$ ODE

...

6.2.5 Incompatibility of Fourier series and Fourier transform methods

Let's assume that we can solve a problem using both methods.

The following hypotheses hold:

- u is piecewise defined
- u is T-periodic
- $\int_{-\infty}^{+\infty} |u(x)|\,dx < \infty$

Then $\int_{-\infty}^{+\infty} |u(x)|\,dx = \sum_{n=-\infty}^{+\infty} \int_{nT}^{(n+1)T} |u(x)|\,dx$

$$= \sum_{n=-\infty}^{+\infty} \int_0^T |u(y+nT)|\,dy = \sum_{n=-\infty}^{+\infty} \underbrace{\int_0^T |u(y)|\,dy}_{I}$$

$y = x - nT$

$u(y) = u(y+nT)$

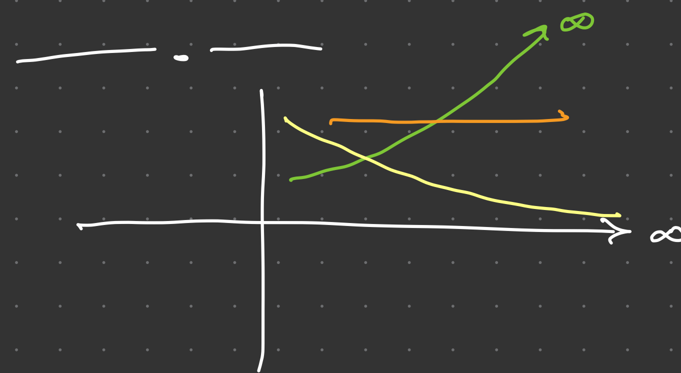
$$\int_{-\infty}^{+\infty} |u(x)|\,dx = \sum_{n=-\infty}^{+\infty} I < \infty \longrightarrow I = 0$$

$$\longrightarrow \int_0^T |u(y)|\,dy = 0 \longrightarrow |u(y)| = 0 \quad \forall\, y \in [0, T]$$

$\longrightarrow \quad u(y) = 0 \quad \forall\, y \in \mathbb{R}.$

u is T-periodic

Then, if both methods are applicable, u is a trivial solution.



$$\int_{-\infty}^{+\infty} |u(x)|\,dx < \infty$$

$$\downarrow$$

$$\lim_{x \to \pm\infty} u(x) = 0.$$

How can $u(x)$ be periodic? $\longrightarrow u(x) = 0.$

6.3 Partial Differential Equations (PDEs)

We are going to solve PDEs using separation of variables.

Then we will use Sturm-Liouville problem.

6.3.1 Heat equation for a finite length bar

Let be $a \in \mathbb{R}^*$, $L \in \mathbb{R}^+$, $f : [0, L] \to \mathbb{R}$, $f \in C^1([0, L])$

s.t. $f(0) = f(L) = 0$. Find the solution $u = (x, t)$ of:

$$\frac{\partial u}{\partial t}(x,t) = a^2 \frac{\partial^2 u}{\partial x^2}(x,t) + f(x) \quad \forall x \in]0, L[\;, \; \forall t > 0$$

$$u(0,t) = u(L,t) = 0 \quad \text{(Dirichlet condition)}$$

$$u(x,0) = u_0(x) \quad \text{(initial condition)}$$