CHAPTER 5: FOURIER TRANSFORM

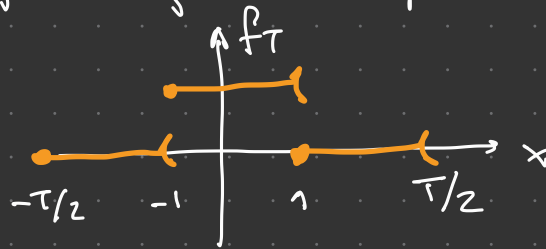
5.1 Introduction

5.1.1 Motivation

Fourier series allow to develop periodic functions as an infinite sum of trigonometric function. Fourier transform allows to study general functions.

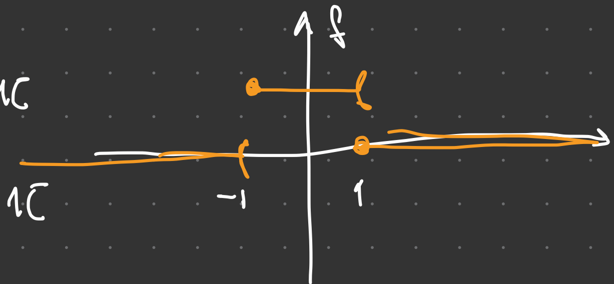
- Idea: let $T > 0$ and f_T be a T-periodic function defined as

$$f_T(x) = \begin{cases} 0 & \text{if } x \in [-T/2, -1[\\ 1 & \text{if } x \in [-1, 1[\\ 0 & x \in [1, T/2[\end{cases}$$



when $T \to \infty$ we have

$$\lim_{T\to\infty} f_T(x) = f(x) = \begin{cases} 1 & \text{if } x \in [-1, 1[\\ 0 & \text{if } x \notin [-1, 1[\end{cases}$$



2.1.2 Heuristic "discovery" of Fourier transform

let $f_T : \mathbb{R} \to \mathbb{R}$ a continuous (for the sake of simplicity) T-periodic s.t. f' is piecewise defined

$$F f_T(y) = \sum_{n=-\infty}^{+\infty} c_n e^{i \frac{2\pi n}{T} y} \quad \forall y \in \mathbb{R}$$

$$c_n = \frac{1}{T} \int_0^T f_T(x) e^{-i \frac{2\pi n}{T} x} dx = \frac{1}{T} \int_{-T/2}^{T/2} f_T(x) e^{-i \frac{2\pi n}{T} x} dx$$

let us introduce $\Delta\alpha = \frac{2\pi}{T}$ and $\alpha_n = \frac{2\pi}{T} n = \Delta\alpha\, n$

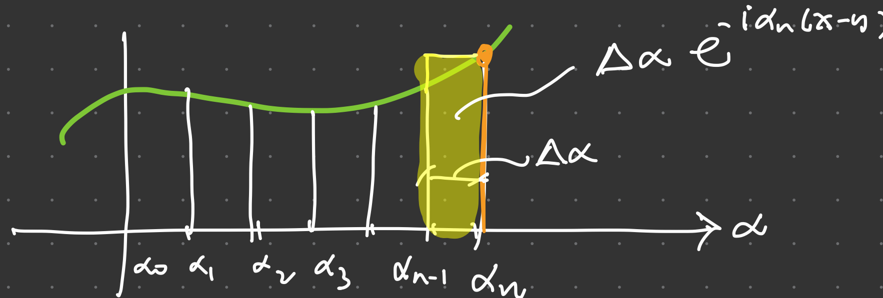
$$= \frac{\Delta\alpha}{2\pi} \int_{-T/2}^{T/2} f_T(x)\, e^{-i\alpha_n x}\, dx$$

$$F f_T(y) = \sum_{n=-\infty}^{+\infty} \left[\frac{\Delta\alpha}{2\pi} \int_{-T/2}^{T/2} f_T(x)\, e^{-i\alpha_n x}\, dx \right] e^{i\alpha_n y}$$

$$= \frac{1}{2\pi} \int_{-T/2}^{T/2} f_T(x) \left[\Delta\alpha \sum_{n=-\infty}^{+\infty} e^{-i\alpha_n (x-y)} \right] dx$$

$$\Delta\alpha = \alpha_n - \alpha_{n-1} = \frac{2\pi}{T} n - \frac{2\pi}{T}(n-1) = \frac{2\pi}{T}$$

Note:



$$(\alpha_n - \alpha_{n-1}) \sum_{n=-\infty}^{+\infty} e^{-i\alpha_n(x-y)}$$ this a Riemann sum

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$$\Delta\alpha \sum_{n=-\infty}^{+\infty} e^{-i\alpha_n(x-y)} = \int_{-\infty}^{+\infty} e^{-i\alpha(x-y)}\, d\alpha$$

$$F f_T(y) = \frac{1}{2\pi} \int_{-T/2}^{T/2} f_T(x) \left[\int_{-\infty}^{+\infty} e^{-i\alpha(x-y)}\, d\alpha \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} f_T(x) e^{-i\alpha x} dx \right]}_{\text{Fourier transform of } f,\ \hat{f},\ \mathcal{F}(f)} e^{i\alpha y} d\alpha$$

Because f_T is continuous, then using Dirichlet thm. $\mathcal{F} f_T(y) = f_T(y)$

$$\lim_{T \to \infty} f_T(y) = \lim_{T \to \infty} \mathcal{F} f_T(y) = f(y)$$

$$\mathcal{F}(f)(\alpha) = \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx$$

$$f(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha) e^{i\alpha y} d\alpha$$

5.2 Fourier transform of a function

5.2.1 Definitions:

let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise-defined s.t.

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < \infty \quad \text{(the integral is bounded)}$$

1) The Fourier transform of f is

$$\mathcal{F}(f) \text{ or } \hat{f} : \mathbb{R} \longrightarrow \mathbb{C}$$
$$\alpha \longmapsto \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\alpha x}\, dx$$

2) The inverse Fourier transform

$$\mathcal{F}^{-1}(f) \text{ or } \hat{f}^{-1} : \mathbb{R} \longrightarrow \mathbb{C}$$
$$x \longmapsto \hat{f}^{-1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\alpha)\, e^{+i\alpha x}\, d\alpha$$

5.2.2 Fourier inversion theorem (reciprocity)

let $f: \mathbb{R} \to \mathbb{C}$ s.t f and f' are piecewise defined

and $\int_{-\infty}^{+\infty} |f(x)| dx < \infty$ (i.e., there exists $\hat{f}$)

and $\int_{-\infty}^{+\infty} |\hat{f}(\alpha)| d\alpha < \infty$

we have:

$$\mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha) e^{i\alpha x} d\alpha = \frac{1}{2}\left(f(x+0) + f(x-0)\right)$$

But, if f is continuous at x, then we have $f(x+0) = f(x-0) = f(x)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha)\, e^{i\alpha x}\, d\alpha.$$

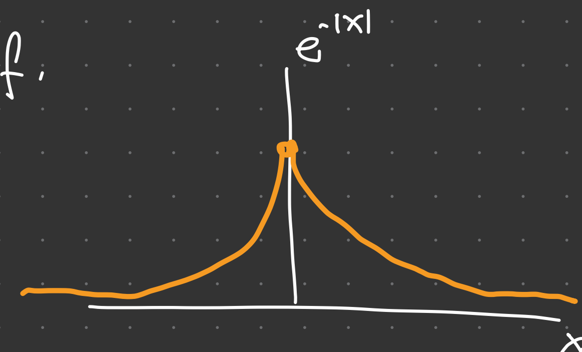
In other words $\mathcal{F}^{-1}(\mathcal{F}(f)) = f$

$$f \xrightarrow{\mathcal{F}} \hat{f} \xrightarrow{\mathcal{F}^{-1}} f$$ (for f continuous at x)

5.2.3 Examples:

- Example 1: Compute the Fourier transform of f.

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto f(x) = e^{-|x|} = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ e^{x} & \text{if } x < 0 \end{cases}$$



$$\int_{-\infty}^{+\infty} |f(x)|\,dx \overset{?}{<} \infty$$

$$\int_{-\infty}^{+\infty} \left| e^{-|x|} \right| dx = \int_{-\infty}^{+\infty} e^{-|x|}\,dx = \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{+\infty} e^{-x}\,dx$$

$$= 2\int_{0}^{+\infty} e^{-x}\,dx = -2e^{-x}\Big|_{0}^{+\infty} = 2 < \infty$$

Note:

$$\int_{-\infty}^{+\infty} |f(x)|\,dx < \infty \longrightarrow f \in L_1 \quad , \quad \int_{-\infty}^{+\infty} |f(x)|^2\,dx < \infty \longrightarrow f \in L_2$$

$$f \in L_\infty \longrightarrow \max_{\mathbb{R}} |f(x)| < \infty$$

About exam

- 2h.
- Same kind of questions (I will provide exams)
- ~ 2/3 of exam are MCQ. (not T/F)
    - wrong answers have negative pts.

~ 1/3 open questions.

- no calculator
- formulaire (1 piece of paper on both sides).
- not all MCQ have same score.

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\alpha x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-|x|} e^{-i\alpha x}\, dx$$

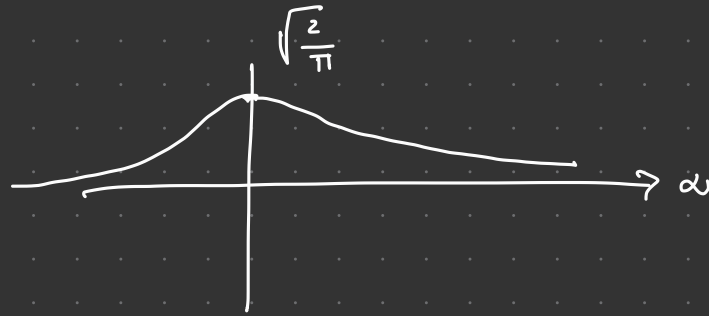
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{x} e^{-i\alpha x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-x} e^{-i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(1-i\alpha)x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-(1+i\alpha)x}\, d\alpha$$

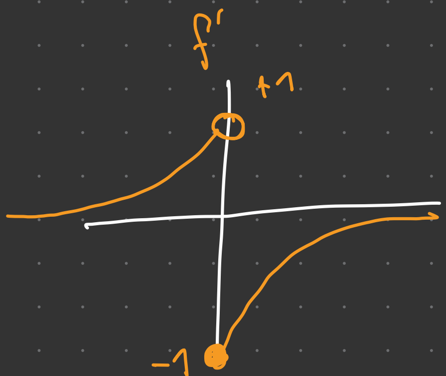
$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(1-i\alpha)x}}{1-i} \right]_{-\infty}^{0} - \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(1+i\alpha)x}}{1+i\alpha} \right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\alpha} - 0 \quad - 0 + \frac{1}{1+i\alpha} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1+i\alpha + 1 - i\alpha}{(1-i\alpha)(1+i\alpha)} = \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2} = \hat{f}(\alpha)$$



$$f'(x) = \begin{cases} -e^{-x} & \text{if } x \geq 0 \\ e^{x} & \text{if } x < 0 \end{cases}$$



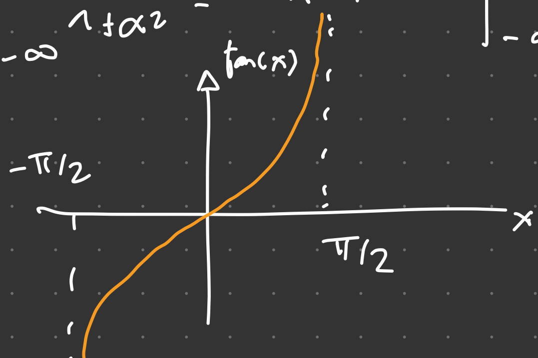
f' is piecewise defined

$$\int_{-\infty}^{+\infty} |\hat{f}(\alpha)| \, d\alpha \overset{?}{<} \infty$$

$$\sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} \left| \frac{1}{1+\alpha^2} \right| d\alpha = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{d\alpha}{1+\alpha^2} \overset{?}{<} \infty$$

$$\frac{d \arctan(\alpha)}{d\alpha} = \frac{1}{1+\alpha^2}$$

$$\int_{-\infty}^{\infty} \frac{d\alpha}{1+\alpha^2} = \arctan \alpha \Big|_{-\infty}^{+\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi < \infty$$



$$\lim_{x \to \pi/2} \tan x = +\infty$$

$$\lim_{x \to -\pi/2} \tan x = -\infty$$

using inversion theorem, (also I have $f \in C^0(\mathbb{R})$)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\alpha) e^{i\alpha x} d\alpha \qquad \forall x \in \mathbb{R}$$

$$f(x) = e^{-|x|} = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} \frac{1}{1+\alpha^2} e^{i\alpha x} d\alpha$$

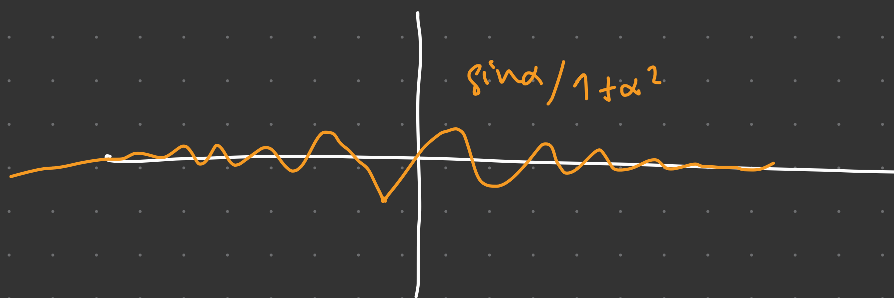
- $x=0 \rightarrow f(0) = 1 = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} \frac{1}{1+\alpha^2} d\alpha$

$$1 = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\alpha}{1+\alpha^2} \rightarrow \int_{-\infty}^{+\infty} \frac{d\alpha}{1+\alpha^2} = \pi \checkmark$$

• $x = 1 \to f(1) = e^{-1} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{e^{i\alpha}}{1+\alpha^2} \, d\alpha$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos\alpha}{1+\alpha^2} \, d\alpha + \underbrace{\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin\alpha}{1+\alpha^2} \, d\alpha}$$

$\Rightarrow$ because $\frac{\sin\alpha}{1+\alpha^2}$ is odd.

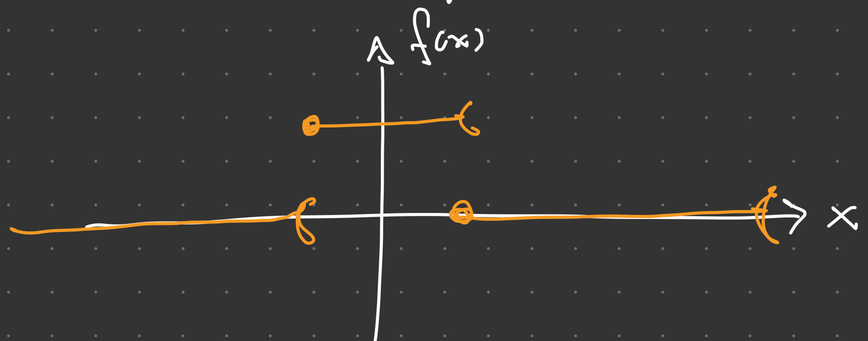


$$e^{-1} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\cos\alpha}{1+\alpha^2} \, d\alpha \longrightarrow \int_{-\infty}^{+\infty} \frac{\cos\alpha}{1+\alpha^2} \, d\alpha = \frac{\pi}{e}$$

- Example 2 : compute the Fourier transform of f:

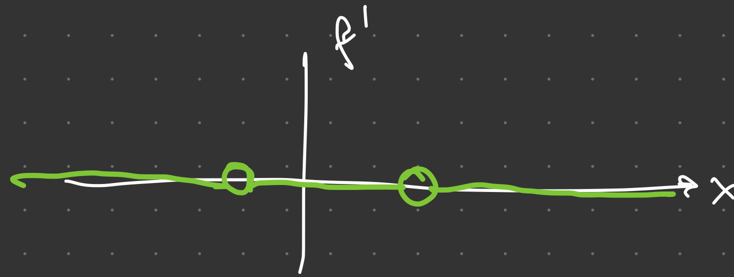
$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$x \longmapsto f(x) = \begin{cases} 1 & \text{if } x \in [-1, 1[\\ 0 & \text{otherwise} \end{cases}$$



f is piecewise defined

f' . . .



$$f \in L_1 \rightarrow \int_{-\infty}^{+\infty} |f(x)|\,dx = \int_{-1}^{+1} 1\,dx = 2 < \infty$$

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\alpha x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} 1\, e^{-i\alpha x}\, dx$$

$$= \frac{-1}{\sqrt{2\pi}} \left. \frac{e^{-i\alpha x}}{i\alpha} \right|_{-1}^{+1} = \frac{-1}{\sqrt{2\pi}} \frac{e^{-i\alpha} - e^{i\alpha}}{i\alpha}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin\alpha}{\alpha} = \hat{f}(\alpha)$$

$$e^{i\alpha} - e^{-i\alpha} = 2i\sin\alpha$$

3 blue 1 brown $\longrightarrow$ Fourier transform $\longrightarrow$ YouTube.