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CHAPTER 4: Fourier Series

4.1 Introduction

4.1.1 Motivation

- Taylor series provides a representation of a function $f: \mathbb{R} \to \mathbb{R}$ infinitely differentiable as a sum of monomials:

$$f(x) = \sum_{n=0}^{\infty} \alpha_n x^n \quad \text{where} \quad \alpha_n = \frac{1}{n!} f^{(n}(0) \quad \text{of Taylor around } x=0.$$

Example: $f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \forall x \in \mathbb{R}.$

- Fourier series : approximate a periodic function as an infinite sum of sines and cosines.

Problem: given $f: \mathbb{R} \to \mathbb{R}$ a T-periodic, we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right) \right]$$

$a_n, b_n \in \mathbb{R}$?

4.1.2 Recalls and preliminary results

- Recall 1: A function $f: \mathbb{R} \to \mathbb{R}$ is T-periodic if $\exists\, T > 0$ s.t.

$$f(x+T) = f(x) \quad \forall\, x \in \mathbb{R}.$$

T is the period of f

Examples: $u_n(x) = \cos\left(\frac{2\pi n}{T} x\right)$ or $v_n(x) = \sin\left(\frac{2\pi n}{T} x\right)$

are $\frac{T}{n}$-periodic for $n \in \mathbb{N}^*$ $\quad (\mathbb{N}^* = \{1, 2, 3, \dots\})$

$$u_n\left(x+\frac{T}{n}\right) = \cos\left(\frac{2\pi n}{T}\left(x+\frac{T}{n}\right)\right) = \cos\left(\frac{2\pi n}{T}x + 2\pi\right)$$

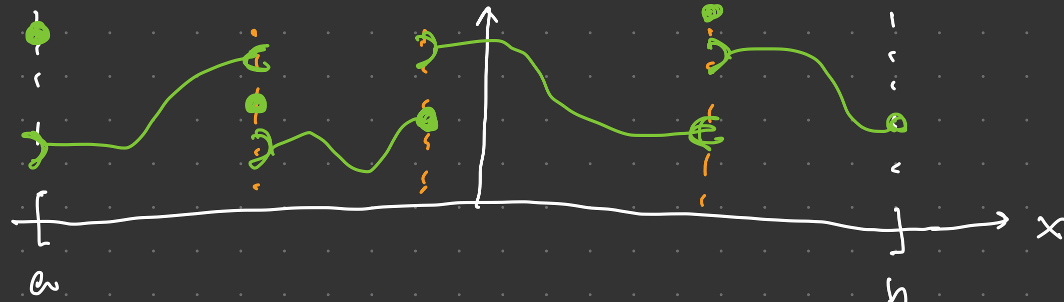
$$= \cos\left(\frac{2\pi n}{T}x\right)$$

The same is true for v_n.

- Recall 2: A function $f: \mathbb{R} \to \mathbb{R}$ is piecewise defined if it has a finite number of discontinuities over every bounded interval. and if at each discontinuity x the limits:

$$\lim_{\substack{t\to x\\ t>x}} f(t) \doteq f(x+0) \quad \text{and} \quad \lim_{\substack{t\to x\\ t<x}} f(t) \doteq f(x-0)$$

exist and are bounded (finite)



- Result 1: let $m, n \in \mathbb{N}^*$ and $T > 0$ then:

a) $$\frac{2}{T}\int_0^T \cos\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi m}{T}x\right)dx$$

$$= \frac{2}{T}\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\sin\left(\frac{2\pi m}{T}x\right)dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

b) $$\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi m}{T}x\right)dx = 0$$

Proof: for the sake of simplicity $T = 2\pi$.

a) $$I = \frac{1}{\pi}\int_0^{2\pi} \cos(nx)\cos(mx) = \frac{1}{2\pi}\int_0^{2\pi}\cos((n-m)x)\,dx$$

$$\cos a \cos b = \frac{1}{2}(\cos(a-b) + \cos(a+b))$$

$$+ \frac{1}{2\pi}\int_0^{2\pi}\cos((n+m)x)\,dx$$

• If $n \neq m$

$$I = \frac{1}{2\pi}\underbrace{\frac{\sin((n-m)x)}{n-m}\Bigg|_0^{2\pi}}_{=0} + \frac{1}{2\pi}\underbrace{\frac{\sin((n+m)x)}{n+m}\Bigg|_0^{2\pi}}_{=0} = 0$$

• If $n = m$

$$I = \frac{1}{2\pi}\overset{=0}{\frac{\sin(0x)}{n-m}\Bigg|_0^{2\pi}} + \frac{1}{2\pi}\underbrace{\frac{\sin(2nx)}{2n}\Bigg|_0^{2\pi}}_{=0}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(0x)\,dx = 1$$

The same for $\frac{2}{T}\int_0^T \sin\left(\frac{2\pi n}{T}x\right)\sin\left(\frac{2\pi m}{T}x\right)dx$

b) $I = \int_0^{2\pi} \sin(nx)\cos(mx)\,dx = \frac{1}{2}\int_0^{2\pi} \sin((n-m)x)\,dx$

$\sin a \cos b = \frac{1}{2}\left(\sin(a-b) + \sin(a+b)\right)$

$$+ \frac{1}{2}\int_0^{2\pi} \sin((n+m)x)\,dx$$

- If $n \neq m$

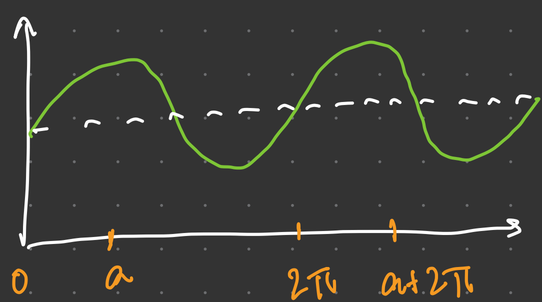
$$I = \frac{1}{2}\left[\underbrace{-\frac{\cos((n-m)x)}{n-m}\Bigg|_0^{2\pi}}_{=0} - \underbrace{\frac{\cos((n+m)x)}{n+m}\Bigg|_0^{2\pi}}_{=0} \right)$$

• If $n=m$

$$I = \frac{1}{2}\int_0^{2\pi} \underbrace{\sin(0)}_{=0}dx + \frac{1}{2}\underbrace{\int_0^{2\pi} \sin((n+m)x)\,dx}_{=0} = 0$$

• Result 2: let $f:\mathbb{R}\to\mathbb{R}$ a T-periodic function, then:

$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx \quad , \quad \forall a\in\mathbb{R}.$$



Proof:

$$\int_a^{a+T} f(x)\,dx = \int_a^0 f(x)\,dx + \int_0^T f(x)\,dx + \int_T^{a+T} f(x)\,dx$$

$$= \int_a^0 f(x)\,dx + \int_0^T f(x)\,dx + \int_0^a f(y+T)\,dy$$

$y = x - T$

$dy = dx$

$$= \cancel{\int_a^0 f(x)\,dx} + \int_0^T f(x)\,dx + \cancel{\int_0^a f(y)\,dy}$$

f is T-periodic

$$= \int_0^T f(x)\,dx \qquad \square$$

4.2 Fourier series of a periodic function

4.2.1 Definitions:

Let $f : \mathbb{R} \to \mathbb{R}$ be a T-periodic piecewise-defined function

For $n \in \mathbb{N}^*$, the partial Fourier series of f and order N is

$$F_N f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right)\right)$$

where a_n and b_n are the Fourier coefficients of f and are given by

$$a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{2\pi n}{T} x\right) dx \qquad n = 0, \dots, N$$

$$b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{2\pi n}{T} x\right) dx \qquad n = 1, \dots, N$$

We call Fourier series of f to the limit

$$Ff(x) = \lim_{N\to\infty} F_N f(x)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n}{T}x\right) + b_n \sin\left(\frac{2\pi n}{T}x\right)\right)$$

when the limit exists...

What is the relationship between $Ff(x)$ and $f(x)$?

4.2.2 Heuristic justification of the definition

For sake the simplicity $T = 2\pi$. Then

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \quad , \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

- $$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)\right)dx$$

$f(x) = Ff(x)$

$$= \int_0^{2\pi}\frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} a_n \underbrace{\int_0^{2\pi}\cos nx\,dx}_{=0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_0^{2\pi}\sin nx\,dx}_{=0}$$

$$= \pi a_0 \rightarrow a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$K \in \mathbb{N}^*$

- $$\int_0^{2\pi} f(x)\cos kx\,dx = \int_0^{2\pi}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)\right)\cos kx\,dx$$

$f(x) = Ff(x)$

$$= \frac{a_0}{2} \underbrace{\int_0^{2\pi} \cos kx\,dx}_{=0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_0^{2\pi} \cos nx \cos kx\,dx}_{= \begin{cases} \pi & \text{if } k=n \\ 0 & \text{if } k \neq n \end{cases}} + \sum_{n=1}^{\infty} b_n \underbrace{\int_0^{2\pi} \sin nx \cos kx\,dx}_{=0}$$

$$= a_k \pi \longrightarrow a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx\,dx \quad \text{for } k \in \mathbb{N}^*$$

- $\int_0^{2\pi} f(x) \sin kx = \dots = \pi b_k \rightarrow b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx\,dx$ for $k \in \mathbb{N}^*$

4.2.3 Fourier series convergence

Questions:

- Which is the relationship between $Ff(x)$ and $f(x)$?
- When is it possible to represent a T-periodic function as an infinite sum of $\cos\left(\frac{2\pi n}{T}x\right)$ and $\sin\left(\frac{2\pi n}{T}x\right)$ which are $\frac{T}{n}$-periodic?

Answer: **Dirichlet theorem**

Let $f: \mathbb{R} \to \mathbb{R}$ be a T-periodic is piecewise-defined then

$$Ff(x) = \lim_{N\to\infty} F_N f(x) \text{ exists } \forall\, x \in \mathbb{R} \text{ and}$$

$$Ff(x) = \frac{1}{2}\left(f(x+0) + f(x-0)\right)$$

$$= \frac{1}{2} \left(\lim_{\substack{t \to x \\ t > x}} f(t) + \lim_{\substack{t \to x \\ t < x}} f(t) \right)$$

In particular, if f is continuous at x, then

$$\underbrace{f(x+0) = f(x-0) = f(x)}_{\text{continuity}} \quad \text{and} \quad \mathcal{F} f(x) = f(x).$$

- Comment: for T-periodic function satisfying the hypotheses of Dirichlet theorem, we can write

$$\frac{1}{2} \left(f(x+0) + f(x-0) \right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n}{T} x \right) + b_n \sin\left(\frac{2\pi n}{T} x \right) \right]$$

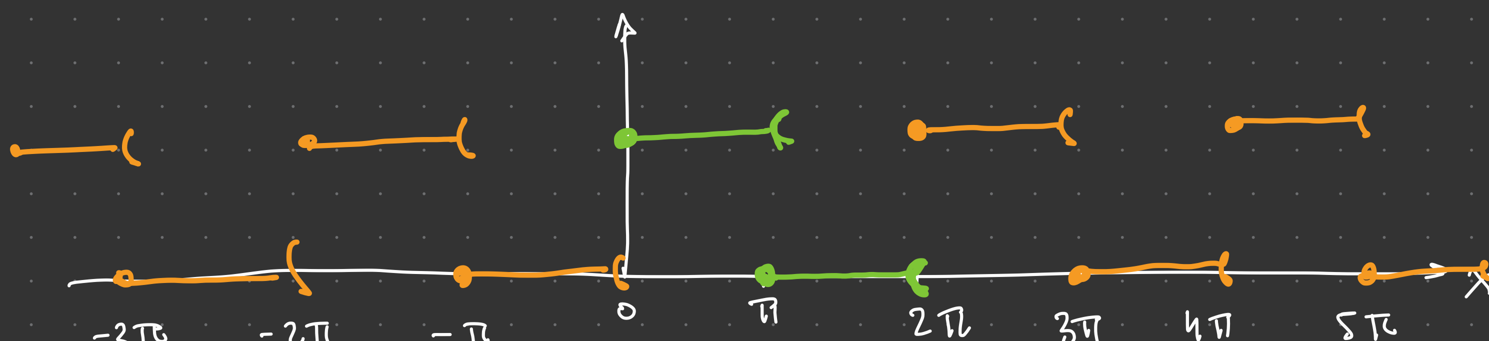
or in other words, f can be represented as a superposition of $u_n = \cos\left(\frac{2\pi n}{T} x \right)$ and $v_n = \sin\left(\frac{2\pi n}{T} x \right)$ that constitutes a basis.

2.2.4 Examples:

- Example 1: let $f : [0, 2\pi[\to \mathbb{R}$ defined as:

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \pi[\\ 0 & \text{if } x \in [\pi, 2\pi[\end{cases}$$

be a function that is extended by 2π-periodicity to $\mathbb{R}$. Compute the Fourier series of f and compare both in $[0, 2\pi]$.



$$F f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) = \frac{1}{\pi} \int_0^{\pi} 1\, dx + \frac{1}{\pi} \int_{\pi}^{2\pi} 0\, dx = 1$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_0^{\pi} 1 \cos nx + \frac{1}{\pi} \int_{\pi}^{0} 0 \cos nx =$$

$$= \frac{1}{\pi} \left. \frac{\sin nx}{n} \right|_0^{\pi} = \frac{1}{\pi} (0 - 0) = 0,$$

$$a_n = 0, \quad \forall n > 0$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_0^{\pi} 1 \cdot \sin nx\, dx + \frac{1}{\pi} \int_{\pi}^{2\pi} 0 \cdot \sin nx\, dx$$

$$= -\frac{1}{n}\,\frac{\cos nx}{n}\Big|_0^{\pi} = \frac{-1}{n\pi}\left(\cos(n\pi) - \cos 0\right) = \frac{-1}{n\pi}\left((-1)^n - 1\right)$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases} ,$$

$$F f(x) = \frac{1}{2} + \sum_{n \text{ odd}} \frac{2}{n\pi} \sin nx = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin\left((2k+1)x\right)}{2k+1}$$