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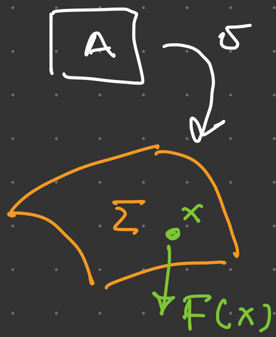
3.2.3 Vector field integrals

- Definition: let $\Sigma \subset \mathbb{R}^3$ be an orientable regular surface parameterized by $\sigma: \bar{A} \to \Sigma$ and let

$$(u,v) \mapsto \sigma(u,v)$$



$$F: \Sigma \to \mathbb{R}^3 \qquad \text{be } F \in C^0(\Sigma, \mathbb{R}^3)$$
$$x \mapsto F(x)$$

The integral of F over Σ in the direction of $\sigma_u \times \sigma_v$ is

$$\iint_\Sigma F \cdot ds = \iint_A F(\sigma(u,v)) \cdot (\sigma_u \times \sigma_v)\, du\, dv$$

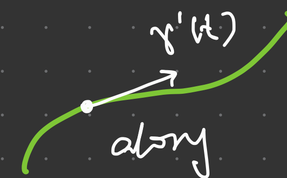
using the unit normal $\nu(u,v) = \dfrac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$

$$\iint_\Sigma F \cdot dS = \iint_A \left[F(\sigma(u,v)) \cdot \nu(u,v) \right] \|\sigma_u \times \sigma_v\| \, du\, dv$$

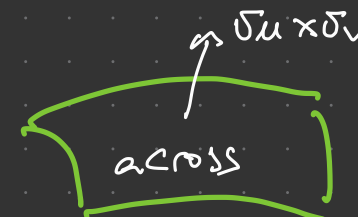
- Remarks:

  - Analogy with integral of a vector field G along Γ

$$\begin{aligned} \gamma : [a,b] &\longrightarrow \Gamma \\ t &\longmapsto \gamma(t) \end{aligned} \qquad \begin{aligned} G : \Gamma &\longrightarrow \mathbb{R}^3 \\ x &\longrightarrow G(x) \end{aligned}$$



$$\int_\Gamma G \cdot dl = \int_a^b G(\gamma(t)) \cdot \gamma'(t)\, dt$$



$$\iint_\Sigma F \cdot dS = \iint_A F(\sigma(u,v)) \cdot (\sigma_u \times \sigma_v)\, du\, dv$$

- For a piecewise regular surface $\Sigma = \bigcup_{i=1}^{k} \Sigma_i$ then:

$$\iint_{\Sigma} F \cdot ds = \sum_{i=1}^{k} \iint_{\Sigma_i} F \cdot ds$$

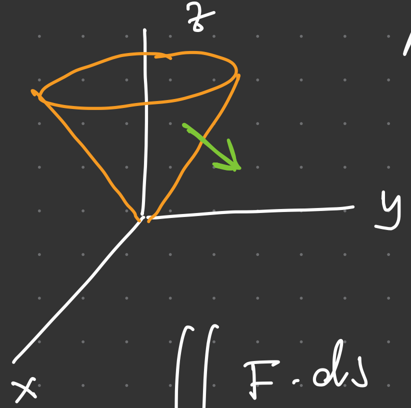
- The integral $\iint_{\Sigma} F \cdot ds$ computes the flux of F through the surface Σ in the direction $\partial u \times \partial v$.

3.2.4 Examples

- Example 1: let $\Sigma = \{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z^2 \text{ and } 0 \leq z \leq 1 \}$ and $F : \Sigma \to \mathbb{R}^3$

$$(x,y,z) \mapsto F(x,y,z) = (y, -x, z^2)$$

Compute $\iint_{\Sigma} F \cdot ds$ across the ascending direction



$$A =]0, 2\pi[\times]0, 1[$$

$$\sigma(\theta, z) = (z\cos\theta, z\sin\theta, z)$$

$$\sigma_\theta \times \sigma_z = \begin{pmatrix} z\cos\theta \\ z\sin\theta \\ -z \end{pmatrix}$$

$$\iint_\Sigma F \cdot dS = -\iint_A F(\sigma(\theta,z)) \cdot (\sigma_\theta \times \sigma_z)\, d\theta\, dz$$

because of ascending direction

$$= \iint_A F(\sigma(\theta,z)) \cdot (\sigma_z \times \sigma_\theta)\, d\theta\, dz$$

$$= -\int_0^{2\pi} \left[\int_0^1 (z\sin\theta, -z\cos\theta, z^2) \cdot (z\cos\theta, z\sin\theta, -z)\, dz \right] d\theta$$

$$= -\int_0^{2\pi} \left[\int_0^1 z^2(\cancel{\sin\theta\cos\theta} - \cancel{\cos\theta\sin\theta} - z)\, dz \right] d\theta$$

$$= \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta = \frac{\pi}{2}.$$

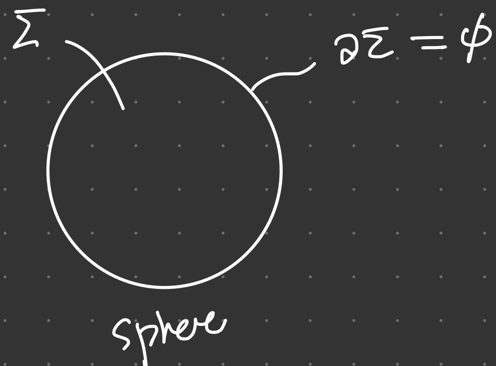
3.3 Divergence theorem

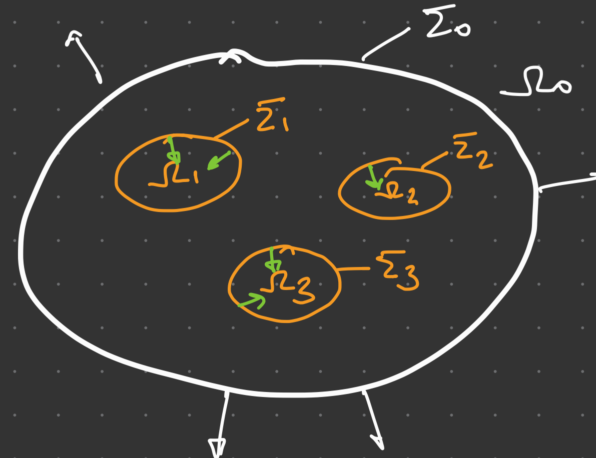
3.3.1 Notations

- Definition: we say that $\Omega \subset \mathbb{R}^3$ is a regular domain if $\exists$ bounded open domains s.t. $\Omega_0, \Omega_1, \dots, \Omega_m \subset \mathbb{R}^3$

  - $\Omega = \Omega_0 \setminus \bigcup_{i=1}^{m} \overline{\Omega}_i$
  - $\overline{\Omega}_j \subset \Omega_0, \ \forall j = 1, \dots, m$
  - $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ if $i \neq j$ and $i, j = 1, \dots, m$
  - $\partial\Omega_j = \Sigma_j, \ j = 0, 1, \dots, m$ where Σ_j are (piecewise) orientable surfaces, and $\partial\Sigma_j = \emptyset$.

Comment about $\partial\Sigma_i = 0$:





3.3.2 Divergence theorem

- Theorem: let $\Omega \subset \mathbb{R}^3$ be a regular domain and $\nu : \partial\Omega \to \mathbb{R}^3$ an outer unit normal vector field. Let $F : \bar{\Omega} \to \mathbb{R}^3$ s.t. $F \in C^1(\bar{\Omega}, \mathbb{R}^3)$ then:

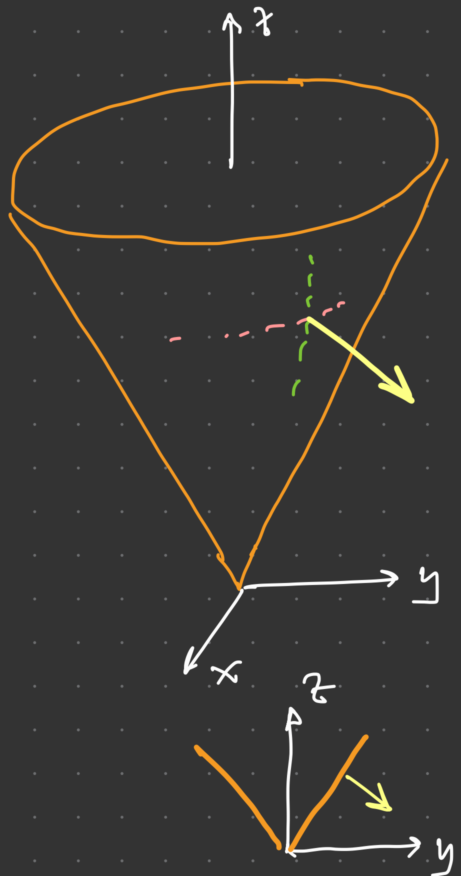
$$\iiint_{\Omega} \operatorname{div} F\,(x,y,z)\,dx\,dy\,dz = \iint_{\partial\Omega} F \cdot \nu\, ds$$

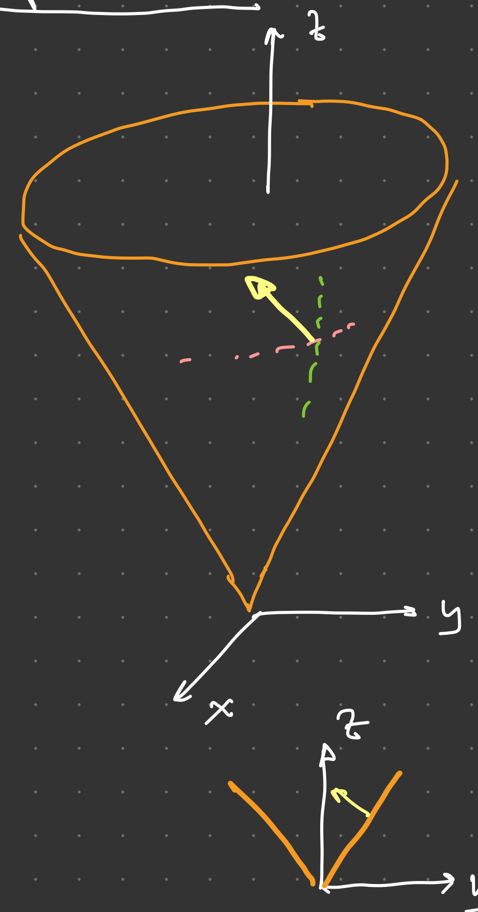
in other words:

$$\iiint_{\Omega} \left(\frac{\partial F^1}{\partial x} + \frac{\partial F^2}{\partial y} + \frac{\partial F^3}{\partial z} \right) dx\,dy\,dz$$

$$= \iint_{\partial\Omega} (F^1 \nu^1 + F^2 \nu^2 + F^3 \nu^3)\, ds$$

Comment about normal direction in example 3.2.4

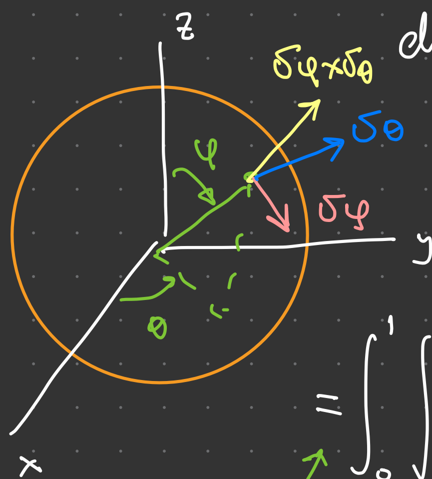




3.3.3 Examples

- Example 1: verify the divergence thm for

$\Omega = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < 1 \}$ and $F(x,y,z) = (xy, y, z)$



$\operatorname{div} F(x,y,z) = y+2$

- $\displaystyle\iiint_\Omega (y+2)\,dx\,dy\,dz$

$$= \int_0^1 \int_0^{2\pi} \int_0^{\pi} (r\sin\varphi\sin\theta + 2)\, r^2 \sin\varphi \, d\varphi\, d\theta\, dr$$

spher. coord.

$$= \cancel{\int_0^1 r^3 dr \underbrace{\int_0^{2\pi} \sin\theta\, d\theta}_{=0} \int_0^{\pi} \sin^2\varphi\, d\varphi} + 2\int_0^1 r^2 dr \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi$$

$$= \frac{4}{3}\pi \left[-\cos\varphi \right]_0^{\pi} = \frac{8\pi}{3}$$

- $\iint_{\partial\Omega} (F \cdot \nu) \, ds$ = integral of scalar field

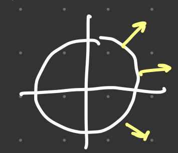
$$\partial\Omega = \Sigma = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1 \}$$

$$\sigma(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

spherical coord.

$$\sigma_\varphi \times \sigma_\theta = \sin\varphi \, \underbrace{(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)}$$

this an outer normal



$$\nu = \frac{\sigma_\varphi \times \sigma_\theta}{\| \sigma_\varphi \times \sigma_\theta \|} \longleftarrow \text{outer unit normal.}$$

$$\iint_{\partial\Omega} (F \cdot \nu)\, ds = \iint_A \underbrace{(F \cdot \nu)(\sigma(\varphi,\theta))}_{g(\sigma(\varphi,\theta))} \| \sigma_\varphi \times \sigma_\theta \| \, d\varphi \, d\theta$$

int. of scalar field

$$(F \cdot \nu)(\sigma(\varphi,\theta)) = F(\sigma(\varphi,\theta)) \cdot \nu(\varphi,\theta)$$

$$\iint_A F(\sigma(\varphi,\theta)) \cdot \frac{\sigma_\varphi \times \sigma_\theta}{\cancel{\| \sigma_\varphi \times \sigma_\theta \|}} \cancel{\| \sigma_\varphi \times \sigma_\theta \|} \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} (\sin^2\varphi\cos\theta\sin\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)\cdot$$

$$(\sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \sin\varphi\cos\varphi)\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} (\sin^4\varphi\cos^2\theta\sin\theta + \sin^3\varphi\sin^2\theta + \sin\varphi\cos^2\varphi)\, d\varphi\, d\theta$$

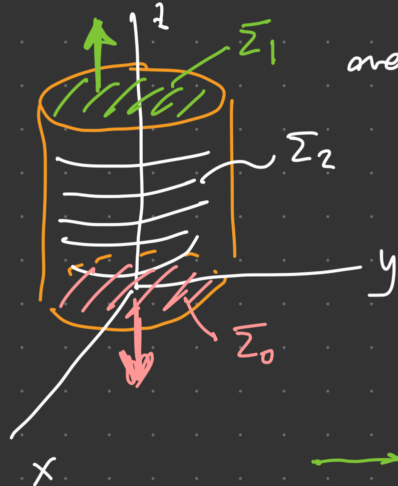
$$= \cancel{\underbrace{\int_0^{2\pi}\cos^2\theta\sin\theta\, d\theta}_{=0}\int_0^{\pi}\sin^4\varphi\, d\varphi} + \underbrace{\int_0^{2\pi}\sin^2\theta\, d\theta}_{=\pi}\ \underbrace{\int_0^{\pi}\sin^3\varphi\, d\varphi}_{=4/3}$$

integr. by parts.

$$+ 2\pi \underbrace{\int_0^{\pi}\sin\varphi\cos^2\varphi\, d\varphi}_{=2/3} = \frac{8\pi}{3}$$

• Example 2: Verify divergence thm for

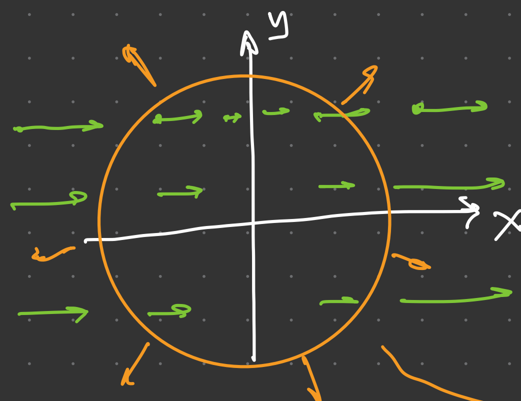
$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2<1 \text{ and } 0<z<1\}$$



and $F(x,y,z) = (x^2, 0, 0)$ $\Rightarrow$ $\operatorname{div} F = 2x$

$$\iiint_\Omega \operatorname{div} F = \iint_{\partial\Omega} F\cdot\nu \, dS$$

$$\partial\Omega = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$$



$$\iint_{\partial\Omega} F\cdot\nu\, dS = \cancel{\iint_{\Sigma_0} F\cdot\nu\, dS} + \cancel{\iint_{\Sigma_1} F\cdot dS} + \cancel{\iint_{\Sigma_2} F\cdot dS}$$

- $\iiint_\Omega \operatorname{div} F(x,y,z)\,dx\,dy\,dz = \iiint_\Omega 2x\,dx\,dy\,dz = \ldots = 0$

cylind. coord

- $\iint_{\partial\Omega} F\cdot\nu\,ds = \iint_{\Sigma_0} F\cdot\nu\,ds + \iint_{\Sigma_1} F\cdot\nu\,ds + \iint_{\Sigma_2} F\cdot ds.$

Parameterize $\Sigma_0, \Sigma_1, \Sigma_2$

Σ_0: $\sigma^0(\theta, r) = (r\cos\theta, r\sin\theta, 0)$

$A^0 =]0, 2\pi[\times]0, 1[$

$\sigma^0_\theta \times \sigma^0_r = \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix}$ pointing down (outer)

Σ_1 : $\sigma^1($ ....

Σ_2 :