23/9/2025

- Theorem 2: let be $\Omega \subset \mathbb{R}^n$ an open domain, and let

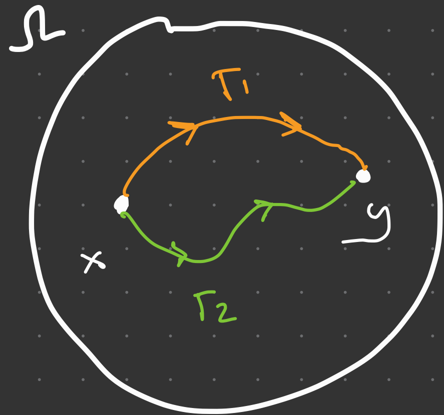
  $F : \Omega \subset \mathbb{R}^n$ be a continuous field $(F \in C^0(\Omega, \mathbb{R}^n))$.

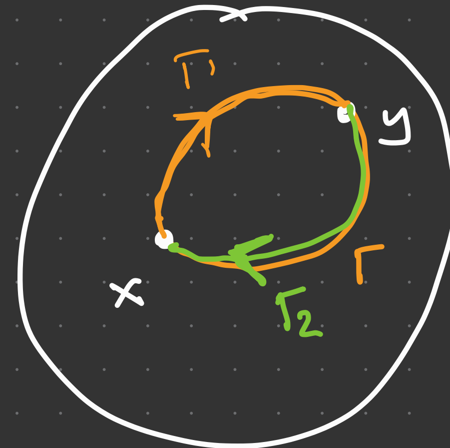
  Then, the following statements are equivalent:

  1) F derives from a potential in Ω

  2) $\int_{\Gamma_1} F \cdot dl = \int_{\Gamma_2} F \cdot dl$ for all the (piecewise) regular simple curves Γ_1 and Γ_2 joining every couple of points x and y, $x, y \in \Omega$.

  3) $\int_\Gamma F \cdot dl = 0$ for every (piecewise) regular simple curve Γ that is closed and $\Gamma \subset \Omega$





$$\int_{\Gamma_1} F\cdot dl = -\int_{\Gamma_2} F\cdot dl$$

$$\int_{\Gamma_1 \cup \Gamma_2} F\cdot dl = 0.$$

- Summary of Theorems 1 and 2: let $\Omega \subset \mathbb{R}^n$ be an open domain

let $F : \Omega \to \mathbb{R}^n$, s.t. $F \in C^1(\Omega, \mathbb{R}^n)$

Theorem 1.

a) If $\text{curl}\, F \neq 0$ in $\Omega \Rightarrow F$ <u>does not</u> derive from a potential

b) If $\text{curl}\, F = 0$ in Ω non-convex and non-simply connected $\Rightarrow$ <u>Theorem 1 says nothing</u>

c) If $\text{curl}\, F = 0$ in Ω convex and/or simply connected

$\Rightarrow$ <u>F derive from a potential</u>

d) If we can find at least one closed (piecewise simple regular curve) $\Gamma \subset \Omega$ s.t. $\int_\Gamma F \cdot d\ell \neq 0$

$\Rightarrow F$ <u>does not derive</u> from a potential

e) If we find one (or more) closed curves $\Gamma \subset \Omega$ s.t.

$$\int_\Gamma F \cdot dl = 0 \Longrightarrow \text{Theorem 2 says nothing.}$$

2.3.2 Examples:

- Example 1:

a) $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$(x,y) \longmapsto F(x,y) = \left(4x^3y^2,\ 2x^4y + y\right)$$

F derives from a potential in $\Omega = \mathbb{R}^2$?

$$\left.\begin{array}{l} \bullet\ \operatorname{curl} F = 0 \Longleftrightarrow \dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x} = 8x^3y \quad \checkmark \\ \bullet\ \Omega = \mathbb{R}^2 : \text{ convex and simply connected.} \quad \checkmark \end{array}\right\} \text{F derives from a potential.}$$

So, let's find the potential...

Find $f \in C^1(\Omega)$ s.t. $\operatorname{grad} f = F$, $f = ?$

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(F_1(x,y), F_2(x,y)\right)$$

$$F_1(x,y) = 4x^3y^2 = \frac{\partial f}{\partial x} \rightarrow \int \frac{\partial f}{\partial x}\,dx = \int 4x^3y^2\,dx$$

$$\rightarrow f(x,y) = x^4y^2 + \alpha(y)$$

$$\frac{\partial f}{\partial y} = F_2(x,y), \quad \frac{\partial f}{\partial y} = \cancel{2x^4y} + \alpha'(y) = F_2(x,y) = \cancel{2x^4y} + y$$

$$\alpha'(y) = y, \quad \int \alpha'(y)\,dy = \int y\,dy \rightarrow \alpha(y) = \frac{1}{2}y^2 + c$$

$c \in \mathbb{R}$

$$f(x,y) = x^4 y^2 + \frac{1}{2} y^2 + C, \quad C \in \mathbb{R}.$$

Alternative:

$$\frac{\partial f}{\partial y} = F_2 = 2x^4 y + \underline{y} \longrightarrow f = x^4 y^2 + \frac{1}{2} y^2 + \beta(x)$$

$$\frac{\partial f}{\partial x} = \cancel{4x^3 y^2} + 0 + \beta'(x) = F_1 = \cancel{4x^3 y^2} \rightarrow \beta'(x) = 0$$

$$\beta(x) = C, \quad C \in \mathbb{R}.$$

$$f(x,y) = x^4 y^2 + \frac{1}{2} y^2 + C.$$

b) $F : \mathbb{R}^3 \to \mathbb{R}^3$

$$(x,y,z) \mapsto F(x,y,z) = \left(y e^x \sin z,\ 1 + e^x \sin z,\ y e^x \cos z + z \right)$$

- $\Omega = \mathbb{R}^3$: convex and simply connected.
- $\operatorname{curl} F = 0$ ✓ (check yourself)

$\Rightarrow$ F derives from a potential.

Let's find $f \in C^1(\Omega)$ s.t. $\operatorname{grad} f = F$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (F_1, F_2, F_3)$$

$$\frac{\partial f}{\partial x} = F_1 \to \int \frac{\partial f}{\partial x} dx = \int F_1 \, dx = \int y e^x \sin z \, dx = \underbrace{y e^x \sin z + \alpha(y,z)}_{f(x,y,z)}$$

$$\frac{\partial f}{\partial y} = F_2 \to \frac{\partial f}{\partial y} = \cancel{e^x \sin z} + \frac{\partial \alpha}{\partial y} = F_2 = 1 + \cancel{e^x \sin z}$$

$$\frac{\partial \alpha}{\partial y} = 1 \longrightarrow \int \frac{\partial \alpha}{\partial y}\, dy = \int 1\, dy \longrightarrow \alpha(y,z) = y + \beta(z)$$

$$\frac{\partial f}{\partial z} = F_3 \quad " \quad \cancel{ye^x \cos z} + \frac{\partial \alpha}{\partial z} = F_3 = \cancel{ye^x \cos z} + z$$

$$\frac{\partial \alpha}{\partial z} = z \quad " \quad 0 + \beta'(z) = z \longrightarrow \beta(z) = \frac{1}{2} z^2 + c \,, \quad c \in \mathbb{R}$$

$$\alpha(y,z) = y + \frac{1}{2} z^2 + c$$

$$\boxed{f(x,y,z) = y e^x \sin z + \frac{1}{2} z^2 + c + y}$$

c) $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x,y,z) \longmapsto F(x,y,z) = (3x^2, 2xz - y, z)$

$\Omega = \mathbb{R}^3$: convex and simply connected

curl $F \neq 0$

$\frac{\partial F_2}{\partial z} = 2x \neq \frac{\partial F_3}{\partial y} = 0 \implies$ curl $F \neq 0 \implies$ F does not derive from a potential.

2.3.4 Proof of Theorem 1 (for $n=2$)

a) If F derives from a potential in Ω and $F \in C^1(\Omega, \mathbb{R}^2)$

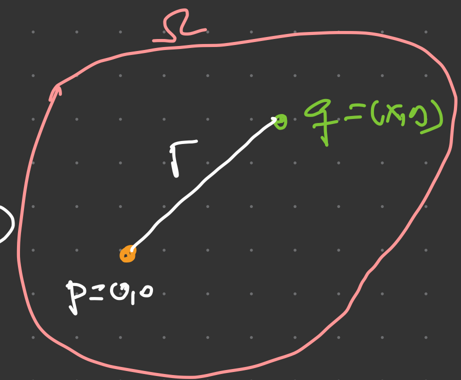
$\Rightarrow$ exists $f \in C^2(\Omega)$ s.t. $\operatorname{grad} f = F$

$\operatorname{curl}(\operatorname{grad} f) = 0 = \operatorname{curl}(F)$ $\square$

b) sufficient condition: (only convex case)

$\operatorname{curl} F = 0 \iff \dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x} \quad \forall\, p, q \in \Omega$ (p and q are points)

I assume that Ω is convex: the arc Γ that joins p and q is s.t. $\Gamma \subset \Omega$



Assume $(0,0) \in \Omega$, $p = (0,0)$, $q = (x, y)$

$\gamma(t) = (tx, ty)$, $t \in [0,1]$

$$\gamma'(t) = (x, y)$$

I want to find $\varphi \in C^2(\Omega)$ s.t. $\operatorname{grad}\varphi = F$

Ansatz: $\varphi = \int_\Gamma F \cdot dl = \int_0^1 F(\gamma(t)) \cdot \gamma'(t)\, dt$

$$= \int_0^1 (F_1(tx, ty), F_2(tx, ty)) \cdot (x, y)\, dt$$

$$= \int_0^1 \big(x F_1(tx, ty) + y F_2(tx, ty)\big)\, dt$$

$\operatorname{curl} F = 0 \Leftrightarrow \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$

$$\operatorname{grad}\varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) = (F_1, F_2) = F$$

$$\frac{\partial \varphi}{\partial x} = \int_0^1 F_1(tx, ty) + tx \frac{\partial F_1}{\partial x}(tx, ty) + ty \frac{\partial F_2}{\partial x}(tx, ty)\, dt$$

$$= \int_0^1 F_1(tx, ty) + tx \frac{\partial F_1}{\partial x}(tx, ty) + ty \frac{\partial F_1}{\partial y}(tx, ty)\, dt$$

$$= \int_0^1 F_1(tx,ty) + t \operatorname{grad} F_1(tx,ty) \cdot \underbrace{(x,y)}_{\gamma'(t)} \, dt$$

$$= \int_0^1 \underbrace{F_1(tx,ty) + t \operatorname{grad} F_1(\gamma(t)) \cdot \gamma'(t)}_{\frac{d}{dt}\left(t\, F_1(\gamma(t))\right)} \, dt$$

$$= \int_0^1 \frac{d}{dt}\left(t\, F_1(\gamma(t))\right) dt = t\, F_1(\gamma(t)) \Big|_0^1$$

$$= 1 \cdot F_1(\gamma(1)) - \cancel{0 \cdot F_1(\gamma(0))} = F_1(\gamma(1)) = F_1(x,y)$$

$$\frac{\partial \varphi}{\partial x} = F_1(x,y).$$

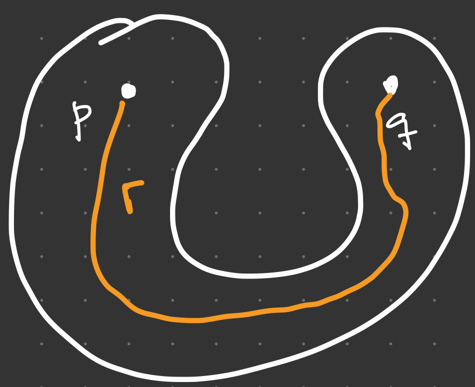
I do the same but for $\frac{\partial \varphi}{\partial y}$ then

$$\frac{\partial \varphi}{\partial y} = F_2(x,y)$$

$$\left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) = \operatorname{grad} \varphi = \left(F_1(x,y), F_2(x,y)\right)$$

F derives from a potential (φ is the potential) $\square$

Note: for simply - connected



2.3.2 Examples (continuation)

Example 2: let be a vector field

$$F : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}^2$$

$$(x,y) \longmapsto F(x,y) = \left(\frac{-y}{x^2+y^2} , \frac{x}{x^2+y^2} \right)$$

and we consider the domains:

$\Omega_1 = \{ (x,y) \in \mathbb{R}^2 : y > 0 \}$, $\Omega_2 = \{ (x,y) \in \mathbb{R}^2 : y < 0 \}$

$\Omega_3 = \mathbb{R}^2 \setminus \{ (x,y) \in \mathbb{R}^2 : \underbrace{x \leq 0 \text{ and } y = 0}_{\text{both conditions should be fulfilled by every point}} \}$, $\Omega_4 = \mathbb{R}^2 \setminus \{(0,0)\}$

Does F derive from a potential in Ω_i , $i = 1,2,3,4$?

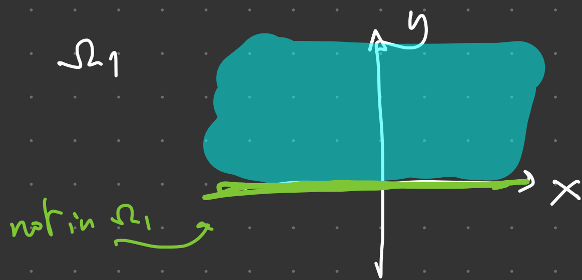
If so, find the potential. otherwise, find a closed curve $\Gamma \subset \Omega_i$

s.t. $\int_\Gamma F \cdot d\ell \neq 0.$

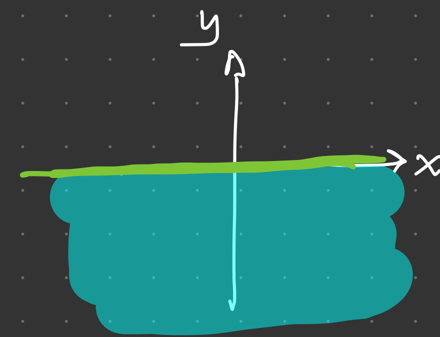
$F \in C^\infty(\mathbb{R}^2 \setminus \{(0,0)\}, \mathbb{R}^2)$

curl $F = 0$? $\rightarrow$ yes! $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$

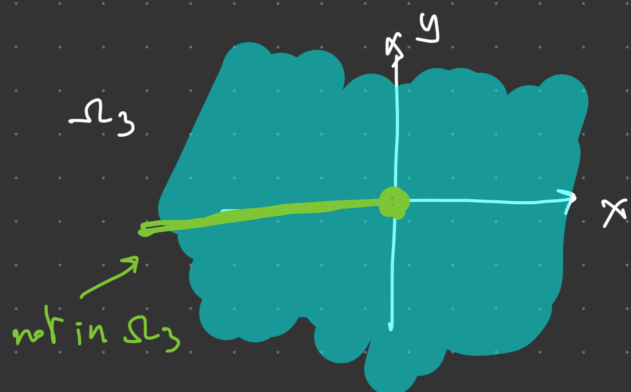
Ω_1



Ω_2



Ω_3



Ω_4