Chapter 1: Differential operators.

1.1. Recalls, notations, and terminology

- $\mathbb{N}$: natural numbers $\{0, 1, 2, 3, \ldots\}$
- $\mathbb{Z}$: integer $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{R}$: real numbers.
- For $n=2$ $\quad (x_1, x_2) = (x, y) \in \mathbb{R}^2$

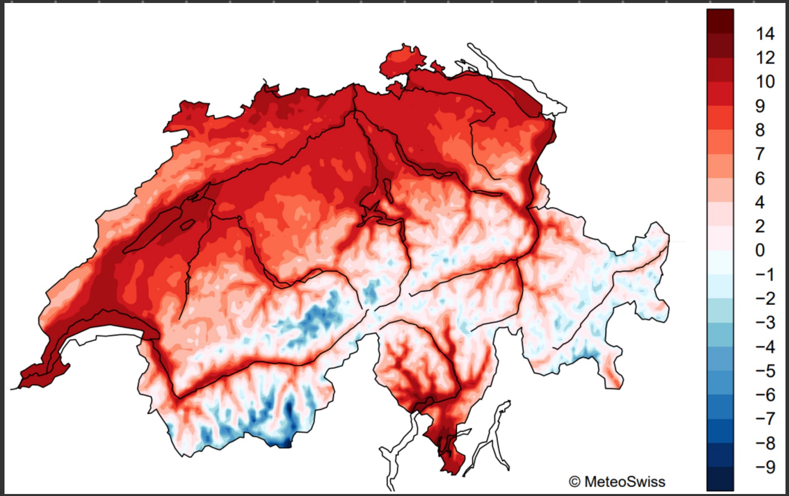
  for $n=3$ $\quad (x_1, x_2, x_3) = (x, y, z) \in \mathbb{R}^3$

  for $n>0$ $\quad x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n\text{ times}}$

- A function that has real values defined on $\Omega \subset \mathbb{R}^n$, $n > 0$

$$f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$x \longmapsto f(x) = f(x_1, x_2, \dots, x_n)$$

that depends on multiple variables $(x_1, \dots, x_n)$ is called a <u>scalar field</u> defined on Ω.



- For $k \in \mathbb{N}$ we write that $f \in C^k(\Omega)$ if all the derivatives with order $\leq k$ exist and are continuous in Ω

Ex $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2} \frac{1}{\sqrt{x}}$

for $k=0 \rightarrow$ the function is continuous in Ω.

- A function that has vector real values and is defined in $\Omega \subset \mathbb{R}^n$

$$F : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad n > 0, \ m > 1$$

$$x \longmapsto F(x) = (F_1(x), F_2(x), \dots, F_m(x))$$

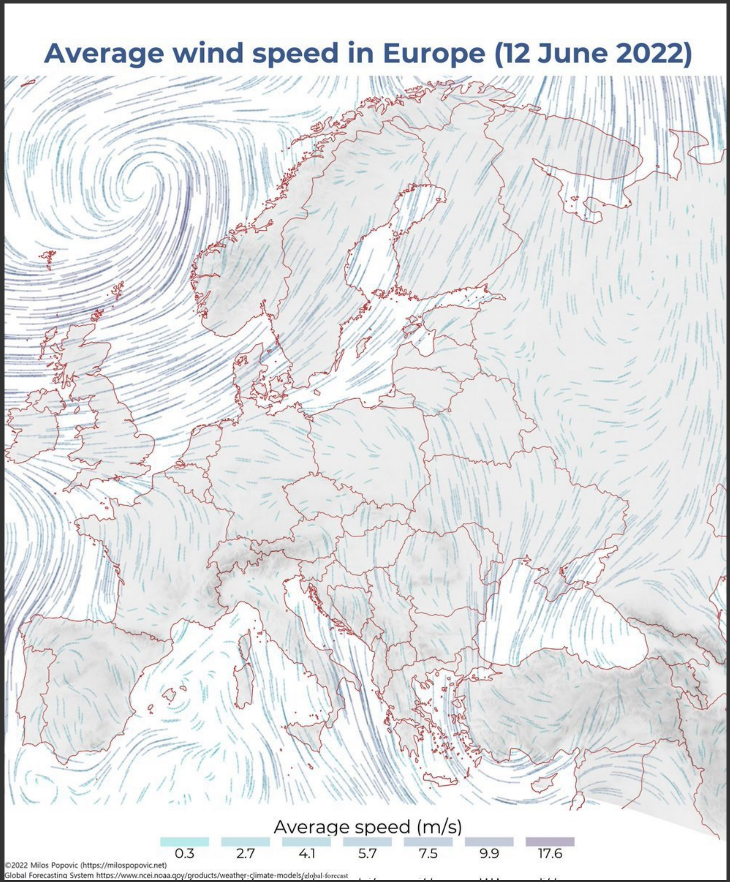
where

$$F_i : \Omega \longrightarrow \mathbb{R}$$

$$x \longmapsto F_i(x) = F_i(x_1, \dots, x_n)$$

is a <u>vector field</u>.

You can see this as a collection of scalar fields.



- For $k \in \mathbb{N}$ we write $F \in C^k(\Omega, \mathbb{R}^m)$ if $F_i \in C^k(\Omega)$, for $i = 1, \dots, m$
- The differential operator nabla, denoted as ∇, is defined by

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

1.2 The gradient operator

1.2.1 Definition

Let be $\Omega \subset \mathbb{R}^n$ $(n \geq 0)$ an open domain and

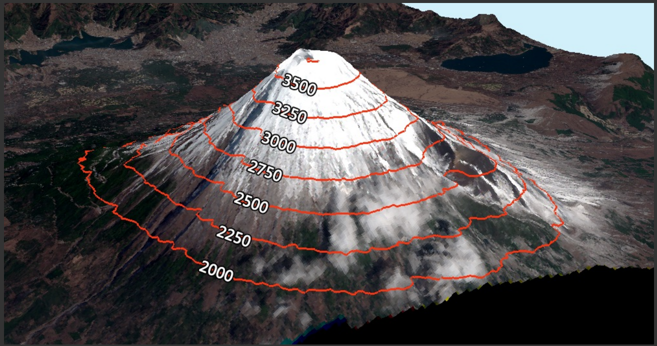
$$f : \Omega \longrightarrow \mathbb{R}$$
$$x \longmapsto f(x) = f(x_1, x_2, \ldots, x_n)$$

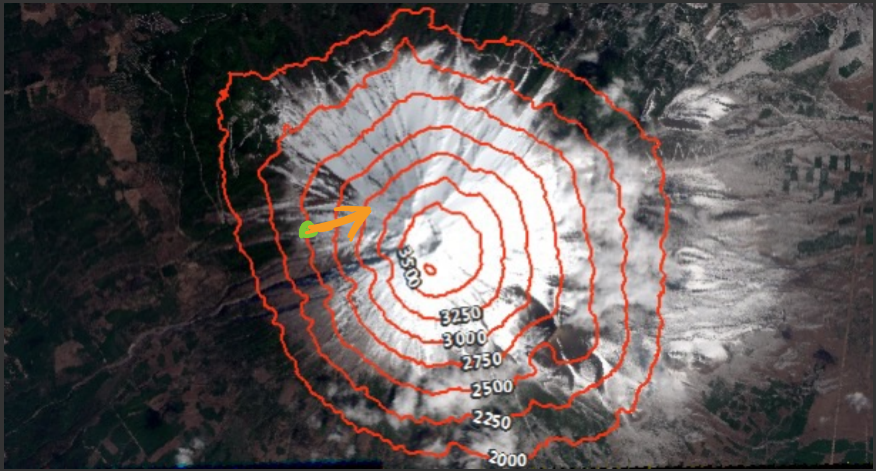
a scalar field $f \in C^1(\Omega)$. The gradient of f is denoted as grad f (or ∇f, or Df) and is defined as

$$\text{grad} f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right)$$

It is the "product" of ∇ and f: $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \cdot f$

2D





scalar field : $\Omega \subset \mathbb{R}^2 \longrightarrow$ height

The gradient of a scalar field is a vector field

$\text{grad} f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Ex: in 3D $\quad f(x) = f(x, y, z) = f(x_1, x_2, x_3)$

$$\text{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

- Example 1: Let f be $f: \mathbb{R}^3 \to \mathbb{R}$

$$(x,y,z) \longmapsto f(x,y,z) = x^2y^3\sin(z^2)$$

$$\operatorname{grad} f(x,y,z) = \left(2xy^3\sin(z^2),\ 3x^2y^2\sin(z^2),\ 2x^2y^3z\cos z^2\right) \in \mathbb{R}^3$$

- Example 2: Gravitational field.

$P_0 = (x_0, y_0, z_0)$

sun M

$m \cdot P = (x, y, z)$

The potential of the gravitational field is

$$f: \mathbb{R}^3 \setminus P_0 \longrightarrow \mathbb{R}$$

$$(x,y,z) \longmapsto f(x,y,z) = \frac{g\,m\,M}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$$

$$r \overset{def}{=} \|P-P_0\| = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$$

$$f(x,y,z) = \varphi(r) = \frac{c}{r}, \quad \text{where } c = gmM$$

$$\operatorname{grad} f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

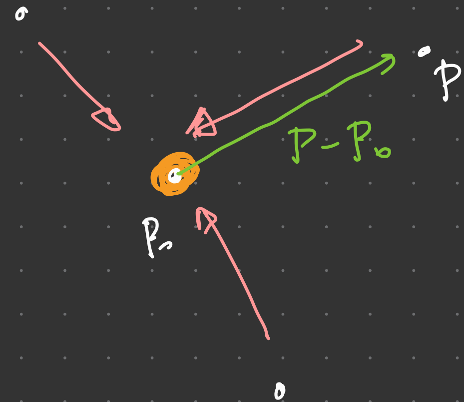
$$\frac{\partial f}{\partial x} = \frac{d\varphi}{dr}\frac{\partial r}{\partial x}, \qquad \frac{d\varphi}{dr} = -\frac{c}{r^2}$$

$$\frac{\partial f}{\partial x} = -\frac{c}{r^2}\frac{x-x_0}{r} \qquad\qquad \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{---}\right) = \frac{x-x_0}{r}$$

$$\frac{\partial f}{\partial y} = -\frac{c}{r^3}(y-y_0)$$

$$\frac{\partial f}{\partial z} = -\frac{c}{r^3}(z-z_0)$$

$$\operatorname{grad} f(x,y,z) = -\frac{c}{r^3}\underbrace{(x-x_0,\ y-y_0,\ z-z_0)}$$



1.3 The divergence operator

let be $\Omega \subset \mathbb{R}^n$ an open domain and

$$F : \Omega \longrightarrow \mathbb{R}^n$$
$$x \longmapsto F(x) \in \mathbb{R}^n$$

s.t. $F \in C^1(\Omega, \mathbb{R}^n)$, the divergence operator of F, denoted as $\operatorname{div} F$ (or $\nabla \cdot F$) is defined by

$$\operatorname{div} F(x) = \frac{\partial F_1}{\partial x_1}(x) + \frac{\partial F_2(x)}{\partial x_2} + \dots + \frac{\partial F_n}{\partial x_n}(x) \quad (\in \mathbb{R})$$

Note: $\nabla \cdot F = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) \cdot (F_1, F_2, \dots, F_n)$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_i$$

Remark: $\operatorname{div} F : \Omega \to \mathbb{R}$ is a scalar field.

Example 1: let be

$$F : \mathbb{R}^3 \to \mathbb{R}^3$$
$$(x, y, z) \longmapsto F(x, y, z) = \left(x^2 - e^y, \sin(xz), y^2 e^{2xz}\right)$$

$$\operatorname{div} F(x, y, z) = \frac{\partial}{\partial x}(x^2 - e^y) + \frac{\partial}{\partial y}(\sin(xz)) + \frac{\partial}{\partial z}(y^2 e^{2xz})$$

$$= 2x + 0 + 2x y^2 e^{2xz} \in \mathbb{R}$$