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CHAPTER 7: DISTRIBUTIONS (Schwartz distributions).

or "on the theory of generalized functions".

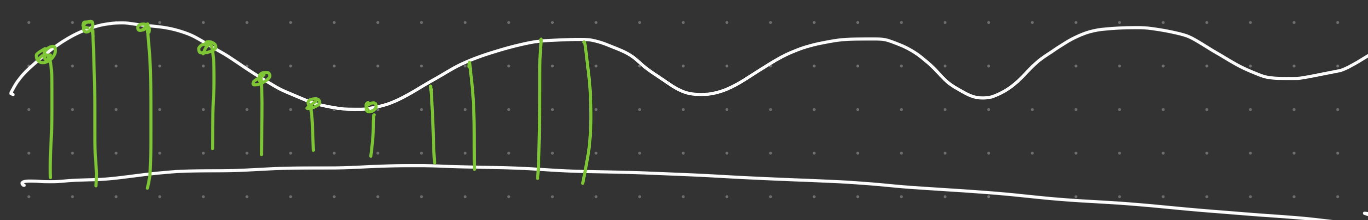
7.1 Introduction

Dirac delta.

$\delta(x)$: it is 0 $\forall \mathbb{R}$ s.t. $x \neq 0$.

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1.$$

Signal



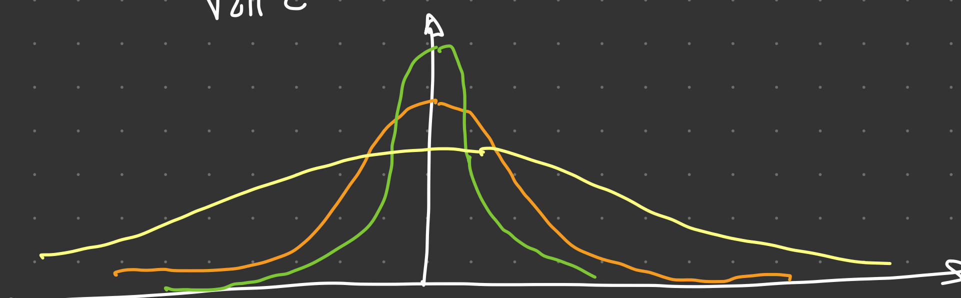
7.1.1 Probing functions

Instead of evaluating f at x_0, we can "probe" it by integrating against a "test function" $\varphi : \mathbb{R} \to \mathbb{R}$

$$\int_{\mathbb{R}} f(x)\,\varphi(x)\,dx$$

Example: Gaussian bump

$$g_\varepsilon(x) := \frac{1}{\sqrt{2\pi}\,\varepsilon} e^{-\frac{x^2}{2\varepsilon^2}} , \quad \varepsilon \in \mathbb{R}^+$$



- g_ε has an integral equal to 1

$$\int_{\mathbb{R}} g_\varepsilon(x)\,dx = \frac{1}{\sqrt{2\pi}\,\varepsilon}\int_{-\infty}^{+\infty} e^{-x^2/2\varepsilon^2} = \frac{\sqrt{2}\,\varepsilon}{\sqrt{2\pi}\,\varepsilon}\underbrace{\int_{-\infty}^{+\infty} e^{-u^2}\,du}_{I}$$

$$u = \frac{x}{\sqrt{2}\,\varepsilon}$$

$$dx = \sqrt{2}\,\varepsilon\,du$$

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-u^2}\,du\right)\left(\int_{-\infty} e^{-v^2}\,dv\right)$$

$$= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} e^{-(u^2+v^2)}\,du\right)dv = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$u^2+v^2 = r^2$$

$$du\,dv = r\,dr\,d\theta$$

$$= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^{+\infty} = 2\pi \left[+0 + \frac{1}{2} \right] = \pi \rightarrow I = \sqrt{\pi}$$

$$\int_{\mathbb{R}} g_\varepsilon(x)\, dx = 1.$$

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} \varphi(x)\, g_\varepsilon(x)\, dx = \varphi(0)$$

Question: Does there exist a function $g_0 : \mathbb{R} \to \mathbb{R}$ with integral 1 s.t. $\int_{\mathbb{R}} g_0(x)\, \varphi(x)\, dx = \varphi(0)$?

Answer: No! But there is a generalize function with this property.

Dirac Delta at zero is a functional

$$\delta_0 : C^0(\mathbb{R}) \to \mathbb{R} \qquad\qquad f : \mathbb{R} \to \mathbb{R}$$

$$f \mapsto f(0).$$

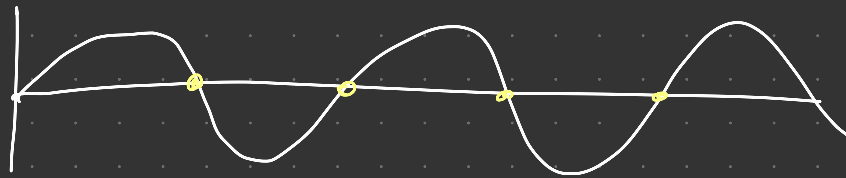
$\xrightarrow{\text{scalar } \mathbb{R}}$ $\boxed{\text{function}}$ $\longrightarrow$ scalar $\mathbb{R}$.

function $\longrightarrow$ $\boxed{\text{functional}}$ $\longrightarrow$ scalar $\mathbb{R}$

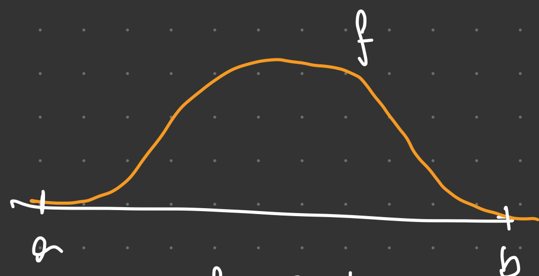
7.2 Definitions

- Support of a function $f : \mathbb{R} \to \mathbb{R}$, denoted $\operatorname{supp} f$, is defined as

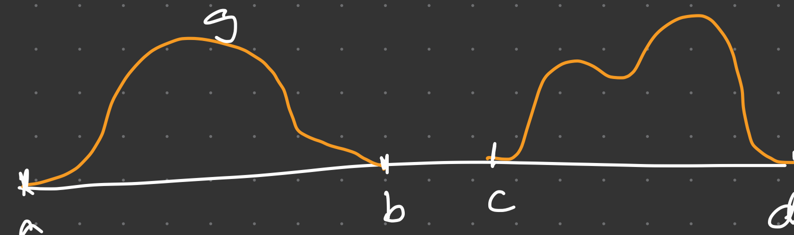
$$\operatorname{supp} f = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$$ closure



$$\operatorname{supp} f = [-1, 1]$$



$$\operatorname{supp} f = [a, b]$$



$$\operatorname{supp} g = [a, b] \cup [c, d]$$

Example: $\varphi : \mathbb{R} \to \mathbb{R}$

$$\varphi(x) = \begin{cases} e^{-1/1-x^2} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$\operatorname{supp} \varphi = [-1, 1]$

$C^0(\mathbb{R})$

- The space of test functions denoted $\mathcal{D}$

If $\varphi : \mathbb{R} \to \mathbb{R}$

$$\mathcal{D} = \{ \varphi \in C^\infty(\mathbb{R}) : \operatorname{supp} \varphi \subseteq \text{bounded interval} \}.$$

The previous φ, $\operatorname{supp} \varphi = [-1, 1]$ (bounded interval) ✓

We check that $\varphi \in C^0(\mathbb{R})$

- $\lim_{x\to 1^-} \varphi(x) = \lim_{x\to 1^-} e^{-1/1-x^2} = 0 = \varphi(1) = \lim_{x\to 1^+} \varphi(x) = 0$
- the same for $x=-1$

$$\varphi'(x) = \begin{cases} \dfrac{2x}{(1-x^2)^2} e^{-1/1-x^2} & \text{if } -1<x<1 \\ 0 & \text{otherwise} \end{cases}$$

you can check continuity:

- $\lim_{x\to 1^-} \varphi'(x) = \lim_{x\to 1^+} \varphi'(x) = \varphi'(1)$
- the same for $x=-1$

you can do it for higher derivatives ...

- Distribution: It is a linear, continuous functional on D.

$T : D \longrightarrow \mathbb{R}$ s.t.

- T is finite
- T is linear

$\forall \alpha, \beta \in \mathbb{R}, \ \forall \varphi, \psi \in D$

$T(\alpha\varphi + \beta\psi) = \alpha T(\varphi) + \beta T(\psi)$

- T is continuous:

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if T changes little, the output changes little.

$\forall \varphi \in D : \operatorname{supp} \varphi \subseteq [a, b]$

$$|T(\varphi)| \leq C \sum_{i=0}^{k} \max_{x \in \mathbb{R}} \left| \frac{\partial^i \varphi(x)}{\partial x^i} \right|$$