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Let's focus on this problem

$$\frac{\partial u}{\partial t}(x,t) = a^2 \frac{\partial^2 u}{\partial x^2}(x,t) \quad x \in]0, L[\,,\quad t > 0$$

$$u(0,t) = u(L,t) = 0 \qquad \text{(Dirichlet condition)}$$

no initial condition.

I assume $u(x,t) = v(x)\,w(t)$ introduce into the PDE

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow v(x)\,w'(t) = a^2\, v''(x)\,w(t)$$

$$u(0,t) = v(0)\,w(t) = u(L,t) = v(L)\,w(t) = 0 \qquad \forall t > 0$$

$$v(0) = v(L) = 0$$

$$v(x)\,w'(t) = a^2\, v''(x)\, w(t) \iff \frac{v''(x)}{v(x)} = \frac{1}{a^2}\frac{w'(t)}{w(t)} \qquad x \in \,]0, L[\quad t > 0$$

$$\frac{v''(x)}{v(x)} = -\lambda = \frac{1}{a^2}\frac{w'(t)}{w(t)}$$

$$\rightarrow \quad \left.\begin{aligned} & v''(x) + \lambda\, v(x) = 0 \\ & v(0) = v(L) = 0 \end{aligned}\right| \longrightarrow \text{Sturm-Liouville.}$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad \forall n \in \mathbb{Z}$$

$$v_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

$$\rightarrow \quad -\left(\frac{n\pi}{L}\right)^2 = \frac{1}{a^2}\frac{w'(t)}{w(t)} \quad \rightarrow \quad w'_n(t) + \left(a\frac{n\pi}{L}\right)^2 w_n(t) = 0$$

$$\omega_n(t) = \exp\left(-\left(\frac{an\pi}{L}\right)^2 t\right)\cdot C \quad \text{where } C \in \mathbb{R}^*$$

$V_n(x)\,\omega_n(t)$ is a solution.

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{L}x\right) e^{-a^2\left(\frac{n\pi}{L}\right)^2 t} C$$

$u(x,0) = f(x) \quad \forall\ x \in]0, L[.$

$$u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{L}x\right) \underbrace{e^{-a^2\left(\frac{n\pi}{L}\right)^2 0}}_{1} C = f(x)$$

$$F_s\left(f(x)\,C^{-1}\right) = f(x)\,C^{-1} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L}x\right)$$

$$b_n = \frac{2}{L} C^{-1} \int_0^L f(x) \sin\left(\frac{\pi n}{L}x\right) dx$$

$$\alpha_n = b_n \ , \quad \beta_n = \alpha_n C = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi n}{L}x\right)dx$$

Let's particularize for $f(x) = 2\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi}{L}x\right)$

$$\beta_1 = \alpha_1 C = 2, \quad \beta_3 = \alpha_3 C = -1, \quad \beta_n = 0 \text{ otherwise}$$

then

$$u(x,t) = 2\sin\left(\frac{\pi}{L}x\right)e^{-a^2\left(\frac{\pi}{L}\right)^2 t} - \sin\left(\frac{3\pi}{L}x\right)e^{-a^2\left(\frac{3\pi}{L}\right)^2 t}$$

6.3.2 Heat equation for an infinite bar

Let be $a \in \mathbb{R}^*$ and $f: \mathbb{R} \to \mathbb{R}$, $f \in C^1(\mathbb{R})$ s.t.

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < \infty \quad \text{and} \quad \int_{-\infty}^{+\infty} |\hat{f}(\alpha)|\, d\alpha < \infty.$$

find the solution $u(x,t)$ of

$$\frac{\partial u}{\partial t}(x,t) = a^2 \frac{\partial^2 u}{\partial x^2}(x,t) \qquad \forall x \in \mathbb{R},\ \forall t > 0.$$

$$u(x,0) = f(x) \qquad \forall x \in \mathbb{R}.$$

Particularize $f(x) = e^{-x^2}$

$$F(u)(\alpha,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(y,t)\, e^{-i\alpha y}\, dy = \hat{u}(\alpha,t)$$

$$F\left(\frac{\partial u}{\partial t}\right)(\alpha,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\partial u}{\partial t}(y,t)\,e^{-i\alpha y}\,dy = \frac{\partial}{\partial t}\left(\hat{u}\right)(\alpha,t)$$

$$F\left(\frac{\partial^2 u}{\partial x}\right)(\alpha,t) = (i\alpha)^2\,\hat{u}(\alpha,t) = -\alpha^2\,\hat{u}(\alpha,t)$$

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t}(\alpha,t) = -a^2\alpha^2\,\hat{u}(\alpha,t), \\ \hat{u}(\alpha,0) = \hat{f}(\alpha) \end{cases}$$ ODE.

$$\hat{u}(\alpha,t) = \exp(-a^2\alpha^2 t)\,c(\alpha)$$

$$\hat{u}(\alpha,0) = \underbrace{\exp(-a^2\alpha^2 0)}_{1}\,c(\alpha) = c(\alpha) = \hat{f}(\alpha)$$

$$\hat{u}(\alpha,t) = \exp(-a^2\alpha^2 t)\,\hat{f}(\alpha).$$

$$f(x) = e^{-x^2} \xrightarrow{\text{Table}} \hat{f}(\alpha) = \frac{1}{\sqrt{2}} e^{-\alpha^2/4}$$

$$\hat{u}(\alpha, t) = \exp(-a^2 \alpha^2 t) \frac{1}{\sqrt{2}} \exp\left(-\alpha^2/4\right)$$