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CHAPTER 6: APPLICATIONS OF FOURIER ANALYSIS TO ODEs AND PDEs

ODE: Ordinary Differential Equation

PDE: Partial Differential Equation.

A system of differential equations, given an interval $]a,b[$, and a function

$$F:]a,b[\times \underbrace{\mathbb{R}^N \times \mathbb{R}^N \times \dots \times \mathbb{R}^N}_{m+1 \text{ times}} \longrightarrow \mathbb{R}^N$$

consists in finding $u:]a,b[\to \mathbb{R}^N$ s.t.

$$F(t, u(t), u'(t), u''(t), \dots, u^{(m)}(t)) = 0$$

Example:

ODE

$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = f(t) \quad \forall t > 0 \qquad a_0, a_1, a_2 \in \mathbb{R}$$

$$y'(t) = \frac{dy}{dt}(t) \qquad f(t) : \mathbb{R}^+ \to \mathbb{R}$$

$$\left(\mathbb{R}^+ = \{ x \in \mathbb{R} \text{ s.t. } x > 0 \} \right)$$

PDE

$$\frac{\partial u}{\partial t}(x,t) = a^2 \frac{\partial^2 u}{\partial x^2}(x,t) \quad \forall x, \forall t > 0 \qquad a \in \mathbb{R}$$

6.2 ordinary differential equation

6.2.1 Cauchy problem

Find the solution $u : \mathbb{R}^+ \to \mathbb{R}$ of the equation

$$a_2 u''(t) + a_1 u'(t) + a_0 u(t) = f(t) \quad \forall t > 0$$

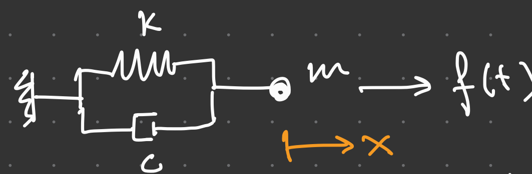
$u(0) = u_0$ and $u'(0) = u_1$ (initial or boundary conditions)

where $a_0, a_1, u_0, u_1 \in \mathbb{R}$ $a_2 \in \mathbb{R}^*$ ($\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\}$)

and $f : \mathbb{R}^+ \to \mathbb{R}$ and a "certain" continuity.

Examples:

- An oscillator



$$m\, x''(t) + c\, x'(t) + k\, x(t) = f(t) \quad \forall t \in \mathbb{R}$$

$$x(0) = x_0, \quad x'(0) = x_1$$

- Population model

$$p'(t) = (\beta - \delta)\, p(t) \quad \forall t > 0$$

$$p(0) = P_0 \quad \text{and} \quad p'(0) = P_1$$

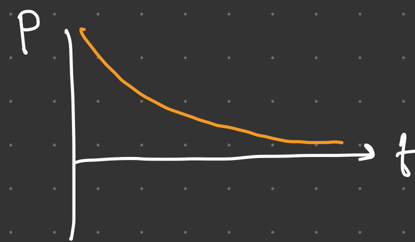
where $p(t)$ is the population (number of individuals)

$\beta, \delta \in \mathbb{R}$

β: is the birth rate

δ: is the death rate

$\beta = 0$, $p'(t) = -\delta\, p(t)$, $p(t) = e^{-\delta t}$



- Euler – Bernoulli beam

$M''(x) = q(x)$

where M is the bending moment

+ boundary condition

6.2.2 Sturm – Liouville problem

let be $u : [0, L] \to \mathbb{R}$ the solution of

$\lambda \in \mathbb{R}$

$$u''(t) + \lambda u(t) = 0 \quad \forall t \in]0, L[$$

$u(0) = u(L) = 0$ (homogeneous Dirichlet conditions)

For which values of $\lambda \in \mathbb{R}$ can we find a non-trivial solution?

- Case 1: $\lambda = 0$, the problem becomes

$$u''(t) = 0 \quad \forall t \in \,]0, L[$$

$$u(0) = u(L) = 0$$

$$\int u'' \, dt = \int 0 \, dt \longrightarrow u'(t) = u_1$$

$$\int u' \, dt = \int u_1 \, dt \longrightarrow u(t) = u_1 t + u_0$$

$$u(0) = 0 = u_1 \cdot 0 + u_0 \longrightarrow u_0 = 0$$

$$u(L) = 0 = u_1 \cdot L + 0 = 0 \longrightarrow u_1 = 0$$

$u(t) = 0 \longrightarrow$ Trivial solution.

- Case 2: $\lambda < 0$

$$u''(t) + \lambda u(t) = 0 \quad \forall t \in]0, L[, \text{ with } \lambda \in \mathbb{R}^-$$

$$u(0) = u(L) = 0$$

Possible solution: $u(t) = \dfrac{u_1}{\sqrt{-\lambda}} \sinh(t\sqrt{-\lambda})$, $u_1 \in \mathbb{R}$

$$u'(t) = \frac{u_1}{\sqrt{-\lambda}} \sqrt{-\lambda} \cosh(t\sqrt{-\lambda}) = u_1 \cosh(t\sqrt{-\lambda})$$

$$u''(t) = \sqrt{-\lambda}\, u_1 \sinh(t\sqrt{-\lambda})$$

$$u''(t) + \lambda u(t) = \sqrt{-\lambda}\, u_1 \sinh(t\sqrt{-\lambda}) + \lambda \frac{u_1}{\sqrt{-\lambda}} \sinh(t\sqrt{-\lambda})$$

$$= \sqrt{-\lambda}\, u_1 \sinh(t\sqrt{-\lambda}) - (-\lambda) \frac{u_1}{\sqrt{-\lambda}} \sinh(t\sqrt{-\lambda}) = 0$$

$\sqrt{-\lambda}$