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5.3 Properties of the Fourier transform

$f, g : \mathbb{R} \to \mathbb{R}$, they are piecewise-defined, s.t.

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < \infty, \quad \int_{-\infty}^{+\infty} |g(x)|\, dx < \infty$$

$$\mathcal{F}(f) = \hat{f} \quad \text{and} \quad \mathcal{F}(g) = \hat{g}$$

5.3.1 Continuity and linearity

- $\hat{f}$ is continuous $\forall \alpha \in \mathbb{R}$ and $\lim_{\alpha \to \pm\infty} \hat{f}(\alpha) = 0$
- $\mathcal{F}(af + bg) = a\,\mathcal{F}(f) + b\,\mathcal{F}(g) \qquad \forall a, b \in \mathbb{R}$

$$= a\hat{f} + b\hat{g}$$

Proof:

$$F(af+bg) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (af+bg) e^{-i\alpha x} dx$$

$$= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f e^{-i\alpha x} dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g e^{-i\alpha x} dx$$

$$= a \hat{f}(\alpha) + b \hat{g}(\alpha) \quad \square$$

5.3.2 Fourier transform of a convolution product

- Definition: the convolution of f and g is a function denoted

$f * g = g * f : \mathbb{R} \to \mathbb{R}$ defined as:

$$(f*g)(x) = \int_{-\infty}^{+\infty} f(x-t) g(t) dt$$

It is symmetric

$$(g*f)(x) = \int_{-\infty}^{+\infty} g(x-t)\, f(t)\, dt = -\int_{+\infty}^{-\infty} g(y)\, f(x-y)\, dy$$

$y = x-t$

$t = x-y$

$-dy = dt$

$$= +\int_{-\infty}^{+\infty} g(y)\, f(x-y)\, dy = \int_{-\infty}^{+\infty} g(t)\, f(x-t)\, dt = (f*g)(x)$$

$y = t$

$dy = dt$

$\square$

$$\mathcal{F}(f*g) = \sqrt{2\pi}\ \mathcal{F}(f)\ \mathcal{F}(g) = \sqrt{2\pi}\ \hat{f}\ \hat{g}$$

Proof:

$$F(f*g)(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (f*g)(x)\, e^{-i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x-t)\, g(t)\, dt \right) e^{-i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x-t)\, e^{-i\alpha x}\, dx \right) g(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(y)\, e^{-i\alpha(y+t)}\, dy \right) g(t)\, dt$$

$y = x - t$

$\cancel{-dy = dt}$

$x = y + t, \quad dx = dy$

$$= \sqrt{2\pi}\ \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)\, e^{-i\alpha y}\, dy}_{\hat{f}(\alpha)}\ \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t)\, e^{-i\alpha t}\, dt}_{\hat{g}(\alpha)}$$

$$= \sqrt{2\pi}\ \hat{f}(\alpha)\ \hat{g}(\alpha) \qquad \square$$

5.3.3 Fourier transform of a derivative

- $f \in C^1(\mathbb{R})$ (i.e. $f' \in C^0(\mathbb{R})$) and $\int_{-\infty}^{+\infty} |f'(x)|\, dx < \infty$ then:

$$\mathcal{F}(f')(\alpha) = i\alpha\, \mathcal{F}(f)(\alpha)$$

If $f \in C^n(\mathbb{R})$ and $\int_{-\infty}^{+\infty} |f^{(k)}(x)|\,dx < \infty$ for $k = 0, 1, \dots, n$

then

$$\mathcal{F}(f^{(k)})(\alpha) = (i\alpha)^k \mathcal{F}(f)(\alpha) \quad \forall\ \alpha \in \mathbb{R} \text{ and } k = 1, \dots, n.$$

Example:

$$u''' + 3u'' + 5u' + u = f(x)$$

$$\mathcal{F}(u''' + 3u'' + 5u' + u)(\alpha) = \mathcal{F}(f)(\alpha)$$

$$\left((i\alpha)^3 + 3(i\alpha)^2 + 5i\alpha + 1\right)\mathcal{F}(u)(\alpha) = \mathcal{F}(f)(\alpha)$$

$$\mathcal{F}(u)(\alpha) = \frac{\mathcal{F}(f)(\alpha)}{(i\alpha)^3 + 3(i\alpha)^2 + 5i\alpha + 1}$$

$$\mathcal{F}^{-1}\big(\mathcal{F}(u)\big)(x) = u(x)$$

Proof:

$$\mathcal{F}(f')(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f'(x)\, e^{-i\alpha x}\, dx$$

$$\left.\begin{aligned} u &= e^{-i\alpha x} \\ v &= f(x) \\ dv &= f'(x)\,dx \\ du &= -i\alpha\, e^{-i\alpha x}\, dx \end{aligned}\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left(e^{-i\alpha x} f(x) \Big|_{-\infty}^{+\infty} + \underbrace{\int_{-\infty}^{+\infty} f(x)\, i\alpha\, e^{-i\alpha x}\, dx}_{\sqrt{2\pi}\, i\alpha\, \hat{f}(\alpha)} \right)$$

$$= (*)$$

$$\lim_{x \to \pm\infty} f(x) e^{i\alpha x} = 0 \iff \lim_{x \to \pm\infty} \left| f(x) e^{-i\alpha x} \right| = \lim_{x \to \pm\infty} |f(x)| = 0$$

comment after lecture.

$$|e^{i\alpha x}| = 1 \ \forall x \in \mathbb{R}$$

because

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < \infty.$$

$$(*) = i\alpha\, \mathcal{F}(f)(\alpha)$$

$\square$

5.3.4 Shift

Note: $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$

- If $a \in \mathbb{R}^*$ $(\mathbb{R}^* = \mathbb{R} \setminus \{0\})$ $\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\}$

$b \in \mathbb{R}$

$$g(x) = e^{-ibx} f(ax) \text{ then:}$$

$$\mathcal{F}(g)(\alpha) = \frac{1}{|a|} \mathcal{F}(f)\left(\frac{\alpha+b}{a}\right) \quad \forall \alpha \in \mathbb{R}$$

Proof: for $b = -\alpha_0$ and $a = 1$ (for sake of simplicity)

$$g(x) = e^{i\alpha_0 x} f(x). \text{ Then:}$$

$$\mathcal{F}(g)(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{i\alpha_0 x}\, e^{-i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i(\alpha-\alpha_0)x}\, dx = \mathcal{F}(f)(\alpha-\alpha_0)$$

$\square$.

$$= \frac{1}{|a|} \hat{f}\left(\frac{\alpha-\alpha_0}{a}\right) \underset{a=1}{=} \hat{f}(\alpha-\alpha_0)$$