13/11/2025

b) Let $f: [0, 2\pi[\longrightarrow \mathbb{R}$ defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \pi[\\ 0 & \text{if } x \in [\pi, 2\pi[\end{cases}$$

extended by 2π-periodicity to $\mathbb{R}$.

$f \notin C^0(\mathbb{R})$.

$$F f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{2k+1}$$

$$\frac{dFf(x)}{dx} = \frac{2}{\pi} \sum_{k=0} \cos((2k+1)x)$$

$$\frac{dFf}{dx}\left(x=\frac{\pi}{2}\right) = \frac{2}{\pi}\sum_{k=0}^{\infty} \cos\left((2k+1)\frac{\pi}{2}\right).$$

$k=0$, $\cos\frac{\pi}{2}=0$

$k=1$, $\cos\frac{3\pi}{2}=0$

$k=2$, $\cos\frac{5\pi}{2}=0$

$\vdots$

$$\frac{dFf}{dx}(x=\pi) = \frac{2}{\pi}\sum_{k=0}^{\infty} \cos\left((2k+1)\pi\right)$$

$k=0$, $\cos\pi=-1$

$k=1$, $\cos 3\pi=-1$

$k=2$, $\cos 5\pi=-1$

$\vdots$

$$= \frac{2}{\pi}\sum_{k=0}^{\infty} -1 \longrightarrow \text{diverges!}$$

- Theorem 2:

let $f: \mathbb{R} \to \mathbb{R}$ be a T-periodic, s.t., f and f' are piecewise defined,

let
$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right)\right),$$
then

$\forall x_0$ and $x \in [0, T]$ we have

$$\int_{x_0}^{x} f(t)\,dt = \int_{x_0}^{x} Ff(t)\,dt = \int_{x_0}^{x} \frac{a_0}{2}\,dt + \sum_{n=1}^{\infty} \int_{x_0}^{x} \left(a_n \cos\left(\frac{2\pi n}{T} t\right) + b_n \sin\left(\frac{2\pi n}{T} t\right)\right) dt$$

1.4 Other Fourier series formulations

on how to build Fourier series of non-periodic functions.

1.4.1 Fourier cosines series

- Theorem

let $f : [0, L] \to \mathbb{R}$ be a continuous function, s.t.

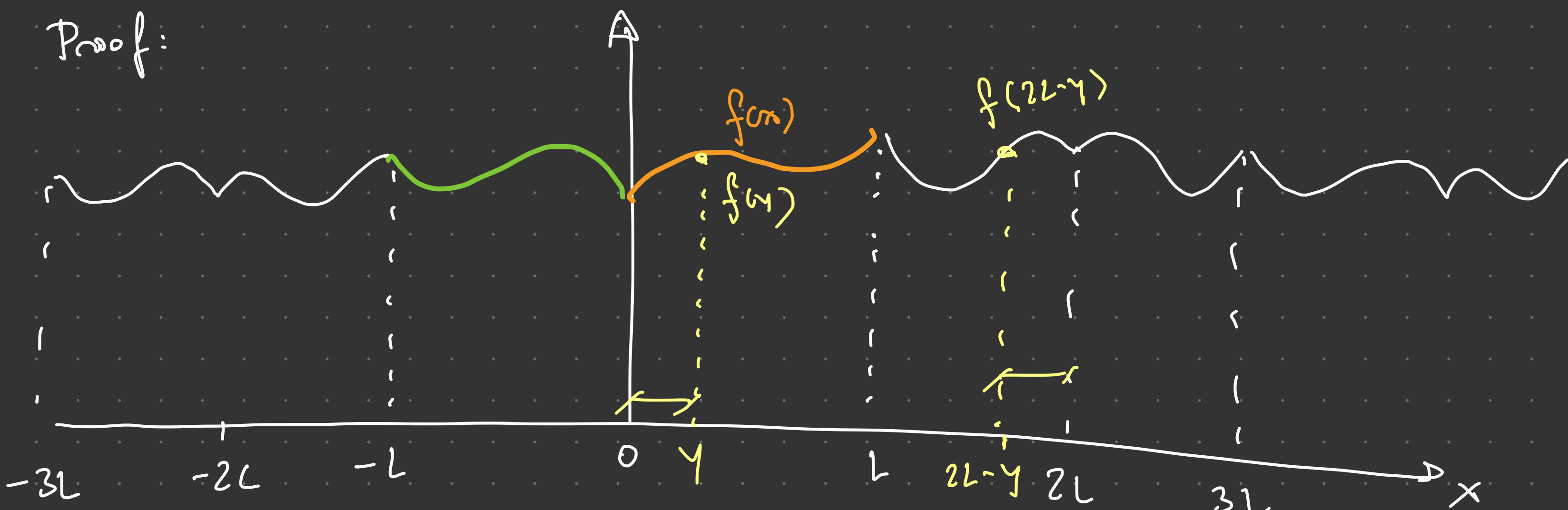
f' is piecewise defined. Then, the series

$$F_c f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n}{L} x\right) \quad \text{with}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{\pi n}{L} x\right) \quad \text{for } n \geq 0 .$$

it is called <u>Fourier cosine series</u> and it converges to f in the interval $[0, L]$. I.e., $f(x) = F_c f(x) \quad \forall x \in [0, L]$

Proof:



$$S(x) = \begin{cases} 1\text{- extending } f(x) \text{ as an even function to } [-L, 0] \\ 2\text{- extending by } 2L\text{-periodicity to } \mathbb{R}. \end{cases}$$

$g : \mathbb{R} \to \mathbb{R}$ is $2L$-periodic and $g \in C^0(\mathbb{R})$, g' is piecewise defined.

$$Fg(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\cancel{2}\pi n}{\cancel{2}L} x\right) \cancel{+ b_n \sin(\quad)}$$

because $g(x)$ is even.

Note: just swapping the integral's limits and changing sign in front would save us the second change of variable

$$a_n = \frac{2}{2L} \int_0^{2L} g(x) \cos\left(\frac{\cancel{2}\pi n}{\cancel{2}L} x\right) dx$$

$$= \frac{1}{L} \int_0^L g(x) \cos\left(\frac{\pi n}{L} x\right) dx + \frac{1}{L} \int_L^{2L} g(x) \cos\left(\frac{\pi n}{L} x\right) dx$$

$$= \frac{1}{L} \int_0^L g(x) \cos\left(\frac{\pi n}{L} x\right) dx - \frac{1}{L} \int_L^0 \underbrace{g(2L-y)}_{g(y)} \underbrace{\cos\left(\frac{\pi n}{L}(2L-y)\right)}_{\cos\left(\frac{n\pi}{L} y\right)} dy$$

$x = 2L - y$

$dx = -dy$

$$= \frac{1}{L} \int_0^L g(x) \cos\left(\frac{\pi n}{L} x\right) dx + \frac{1}{L} \int_{-L}^0 \underbrace{g(-t)}_{g(t)} \underbrace{\cos\left(-\frac{n\pi}{L} t\right)}_{\cos\left(\frac{n\pi}{L} t\right)} dt$$

$y = -t$

$dy = -dt$

$$= \frac{1}{L}\int_0^L g(x)\cos\left(\frac{\pi n}{L}x\right)dx + \frac{1}{L}\int_0^L g(t)\cos\left(\frac{\pi n}{L}t\right)dt$$

$$= \frac{2}{L}\int_0^L g(x)\cos\left(\frac{\pi n}{L}x\right)dx = \underbrace{\frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi n}{L}x\right)dx}_{\text{coefficient of } F_c f(x).} \qquad \square$$

1.4.2 Fourier sine series

- Theorem:

Let $f : [0, L] \to \mathbb{R}$ be a continuous function. s.t.

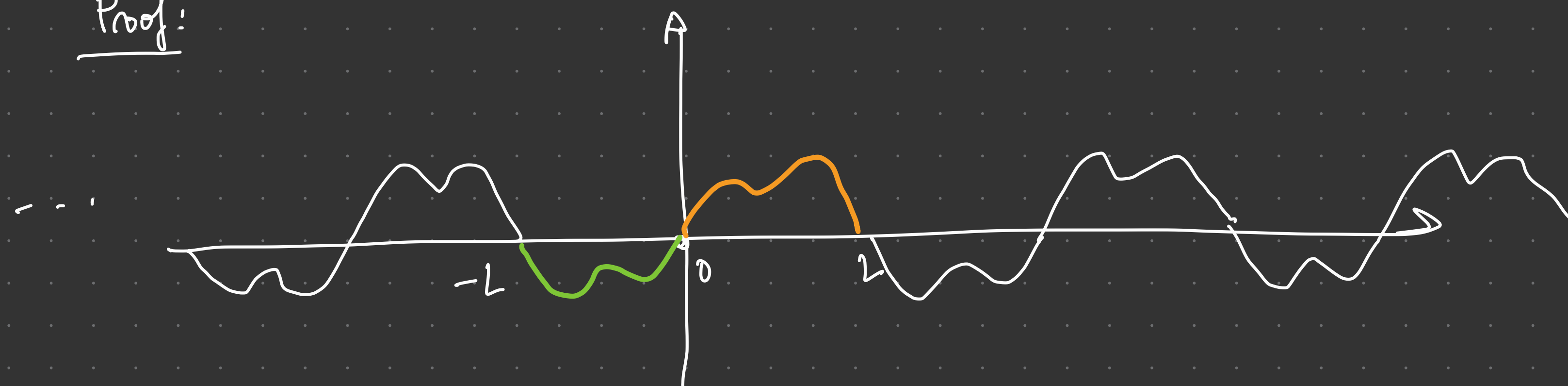
$f(0) = f(L) = 0$ and f' is piecewise defined. Then

$$F_s f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L} x\right) \text{ with}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n}{L} x\right) dx \quad \forall\, n \geq 1.$$

denoted as Fourier sine series of f and it converges to f in $[0, L]$. So, $f(x) = F_s f(x) \quad \forall\, x \in [0, L]$

<u>Proof:</u>



$$g(x) = \begin{cases} 1\text{- extension of } f(x) \text{ as an odd function to } [-L,0] \\ 2\text{- extension by } 2L\text{-periodicity to } \mathbb{R}. \end{cases}$$

$g \in C^0(\mathbb{R})$, g is $2L$-periodic and g is an odd function.

$$Fg(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\cancel{2}\pi n}{\cancel{2}L} x\right)$$

$$b_n = \frac{\cancel{2}}{\cancel{2}L} \int_0^{2L} g(x) \sin\left(\frac{\cancel{2}\pi n}{\cancel{2}L} x\right) dx =$$

$$= \frac{1}{2} \int_0^{L} g(x) \sin\left(\frac{\pi n}{2} x\right) dx + \frac{1}{2} \int_L^{2L} g(x) \sin\left(\frac{\pi n}{2} x\right) dx$$

$y = -x$

$-dy = dx$

$$= \frac{1}{2}\int_0^L g(x)\sin\left(\frac{\pi n}{2}x\right)dx - \frac{1}{2}\int_0^{-L} \underbrace{g(-y)}_{-g(y)}\,\underbrace{\sin\left(-\frac{\pi n}{2}y\right)}_{-\sin\left(\frac{\pi n}{L}y\right)}dy$$

$$= \frac{1}{2}\int_0^L g(x)\sin\left(\frac{\pi n}{2}x\right)dx - \frac{1}{L}\int_0^{-L} g(y)\sin\left(\frac{\pi n}{2}y\right)dy$$

$$= \dots$$

$$b_n = 2\,\frac{1}{2}\int_0^L g(x)\sin\left(\frac{\pi n}{L}x\right)dx$$

Then $\forall\, x \in [0, L]$ $\quad f(x) = g(x) = F_g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L}x\right)$

$$= F_s f(x) \qquad \square.$$