$$= -\frac{1}{n}\frac{\cos nx}{n}\Big|_0^{\pi} = \frac{-1}{n\pi}\left(\cos(n\pi) - \cos 0\right) = \frac{-1}{n\pi}\left((-1)^n - 1\right)$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases},$$

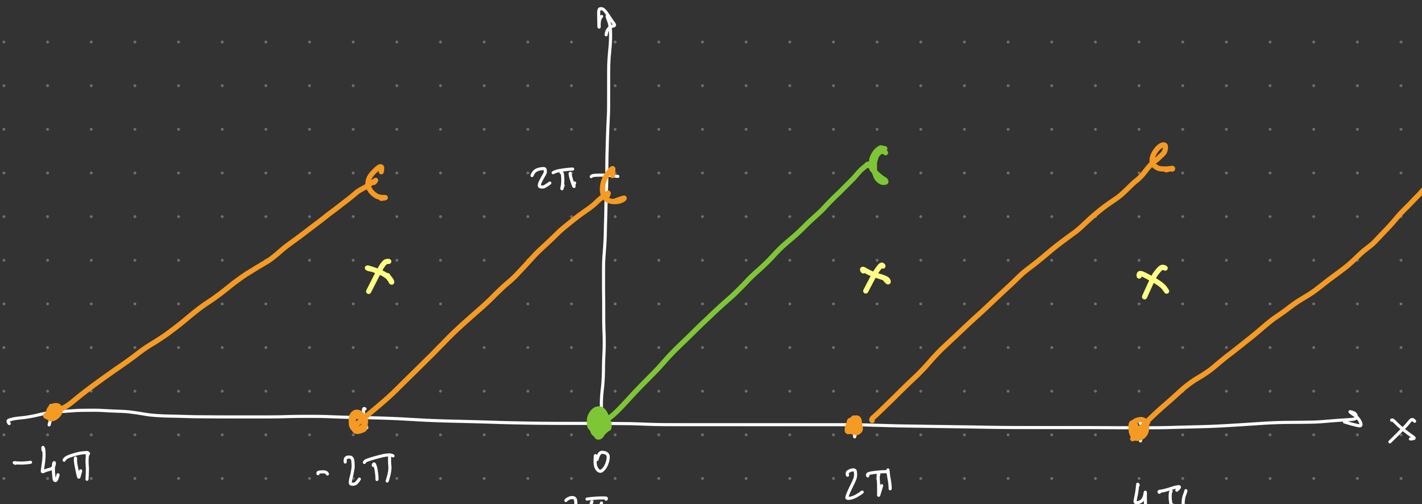
$$Ff(x) = \frac{1}{2} + \sum_{n \text{ odd}} \frac{2}{n\pi}\sin nx = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)x)}{2k+1}$$

6/11/2025.

Dirichlet

$$Ff(x) = \begin{cases} 1/2 = \frac{1}{2}\left(f(0-0) + f(0+0)\right) = \frac{1}{2}(0+1) = \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{2}(1+1) = 1 & \text{if } x \in \,]0,\pi[\\ \frac{1}{2}(1+0) = 1/2 & \text{if } x = \pi \\ \frac{1}{2}(0+0) = 0 & \text{if } x \in \,]\pi, 2\pi[\\ \frac{1}{2}(0+1) = 1/2 & \text{if } x = 2\pi \end{cases}$$

Example 2: same as example 1, but with $f : [0, 2\pi[\to \mathbb{R}$ with $f(x) = x$, extended by 2π-periodicity to $\mathbb{R}$



$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} x\,dx = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \underset{u}{x}\,\overbrace{\cos nx\,dx}^{dv} =$$

$$= \frac{1}{\pi} x \frac{\sin nx}{n}\Big|_0^{2\pi} - \frac{1}{n\pi}\underbrace{\int_0^{2\pi} \sin nx\, dx}_{=0} = 0$$

by parts

$u = x$

$dv = \cos nx\, dx$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin nx\, dx$$

$$= -\frac{1}{\pi} x \frac{\cos nx}{n}\Big|_0^{2\pi} + \frac{1}{n\pi}\underbrace{\int_0^{2\pi} \cos nx\, dx}_{=0} = -\frac{1}{\pi}\frac{2\pi}{n} = -\frac{2}{n}$$

$u = x$

$dv = \sin nx\, dx$

$$Ff(x) = \pi - \sum_{n=1}^{\infty} \frac{2}{n}\sin nx$$

Dirichlet

$$Ff(x) = \begin{cases} \pi = \frac{1}{2}(2\pi + 0) = \pi & \text{if } x = 0 \\ \frac{1}{2}(x + x) = x = f(x) & \text{if } x \in]0, 2\pi[\ (x \in (0, 2\pi)) \\ \pi = \frac{1}{2}(0 + 2\pi) = \pi & \text{if } x = 2\pi \end{cases}$$

1.2.5 Complex notation for Fourier Series

The Fourier series of a T-periodic piecewise-defined function $f : \mathbb{R} \to \mathbb{R}$ in complex form is

$$Ff(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i \frac{2\pi n}{T} x} \quad \text{where } c_n \in \mathbb{C}$$

and $$c_n = \frac{1}{T} \int_0^T f(x) e^{-i \frac{2\pi n}{T} x} dx$$

For the sake of simplicity $T = 2\pi$

$$Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right), \quad \sin nx = \frac{1}{2} i \left(e^{-inx} - e^{inx}\right)$$

missing in the lecture.