16/10/2025.

Σ_1 : $\sigma^1(\theta, r) = (r\cos\theta, r\sin\theta, 1)$

$\Delta^1 =]0, 2\pi[\times]0, 1[$

$\sigma^1_\theta \times \sigma^1_r = \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix}$ pointing down (inner)

Σ_2 : $\sigma^2(\theta, z) = (\cos\theta, \sin\theta, z)$

$\Delta^2 =]0, 2\pi[\times]0, 1[$

$\sigma^2_\theta \times \sigma^2_z = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$ outer normal.

$$\iint_{\partial\Omega} F \cdot \nu \, ds = \iint_{\Sigma_0} F \cdot \nu \, ds + \iint_{\Sigma_1} F \cdot \nu \, ds + \iint_{\Sigma_2} F \cdot \nu \, ds$$

$$\iint_{\Sigma_0} F\cdot\nu\, ds = \iint_{A^0} (F\cdot\nu)(\sigma^0(\theta,r))\, \|\sigma^0_\theta \times \sigma^0_r\|\, d\theta\, dr$$

$$= \int_0^{2\pi}\int_0^1 F(\sigma^0(\theta,r)) \cdot \frac{\sigma^0_\theta \times \sigma^0_r}{\cancel{\|\sigma^0_\theta \times \sigma^0_r\|}} \cancel{\|\sigma^0_\theta \times \sigma^0_r\|}\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta, 0, 0)\cdot(0,0,-r)\, dr\, d\theta = 0$$

because normal is inner

$$\iint_{\Sigma_1} F\cdot\nu\, ds = \dots =$$

$$- \int_0^{2\pi}\int_0^1 F(\sigma^1(\theta,r)) \cdot \frac{\sigma^1_\theta \times \sigma^1_r}{\cancel{\|\sigma^1_\theta \times \sigma^1_r\|}} \cancel{\|\sigma^1_\theta \times \sigma^1_r\|}\, dr\, d\theta$$

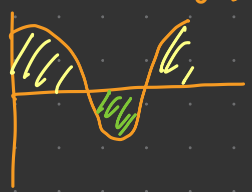
$= \dots = 0.$

$$\iint_{\Sigma_2} F \cdot \nu \, ds = \int_0^{2\pi} \int_0^1 F(\sigma^2(\theta, z)) \cdot (\sigma_\theta^2 \times \sigma_z^2) \, dz \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (\cos^2\theta, 0, 0) \cdot (\cos\theta, \sin\theta, 0) \, dz \, d\theta$$

$$= 1 \cdot \int_0^{2\pi} \cos^3\theta \, d\theta = 0.$$

Note



3.3.4 Corollary and application

- Corollary: If the domain $\Omega \subset \mathbb{R}^3$ and the unit outer normal field $\nu : \partial\Omega \to \mathbb{R}^3$ then

$$\mathrm{Vol}(\Omega) = \frac{1}{3} \iint_{\partial\Omega} (F \cdot \nu)\, ds = \iint_{\partial\Omega} (G_i \cdot \nu)\, ds \quad , \; i = 1, 2, 3$$

where $F(x,y,z) = (x,y,z)$ (s.t. $\operatorname{div} F = 3$),
$G_1(x,y,z) = (x,0,0)$, $G_2(x,y,z) = (0,y,0)$, $G_3(x,y,z) = (0,0,z)$

$\operatorname{div} G_i = 1$, $i = 1, 2, 3$

Div thm

Proof: $\mathrm{Vol}(\Omega) = \iiint_{\Omega} 1\, dv = \iint_{\partial\Omega} (G_i \cdot \nu)\, ds = \frac{1}{3} \iint_{\partial\Omega} F \cdot \nu\, ds.$

- Practical remark:

$$\sigma : A \longrightarrow \partial\Omega$$
$$(u,v) \longmapsto \sigma(u,v) = (\sigma^1(u,v), \sigma^2(u,v), \sigma^3(u,v))$$

$$\mathrm{vol}(\Omega) = \iint_A \sigma^1(u,v)\left(\sigma^2_u \sigma^3_v - \sigma^3_u \sigma^2_v\right) du\,dv$$

$$\sigma_u \times \sigma_v = \begin{pmatrix} \sigma^2_u \sigma^3_v - \sigma^3_u \sigma^2_v \\ \vdots \\ \vdots \end{pmatrix}$$

- Application: continuity equation

Let $v : \Omega \to \mathbb{R}^3$ the flow velocity of a fluid at a point $x \in \Omega$ at the instant t.

Let $\rho(x,y,t)$ be the density of the fluid at $x \in \Omega$ and at instant t.

Mass of the fluid in Ω $\quad$ mass $= \iiint_\Omega \rho \, dv$

$$\iint_{\partial\Omega} \rho v \cdot \nu = - \frac{\partial \, \text{mass}}{\partial t}$$

Div theorem

$$\|$$

$$\iiint_\Omega \operatorname{div}(\rho v) \, dv = - \frac{\partial}{\partial t} \iiint_\Omega \rho \, dv$$

$$\operatorname{div}(\rho v) = - \frac{\partial}{\partial t} \rho \longrightarrow \operatorname{div}(\rho v) + \frac{\partial}{\partial t} \rho = 0$$

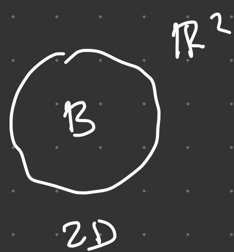
3.4 Stokes' theorem (or Kelvin-Stokes)

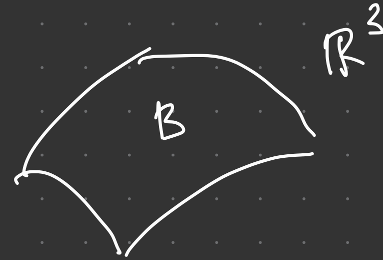
3.4.1 Motivation

- Chapter 2: Green's theorem

$$\iint_B \operatorname{curl} G(x,y)\, dx dy = \int_{\partial B} G \cdot dl \quad \text{for } B \subset \mathbb{R}^2 \text{ a}$$

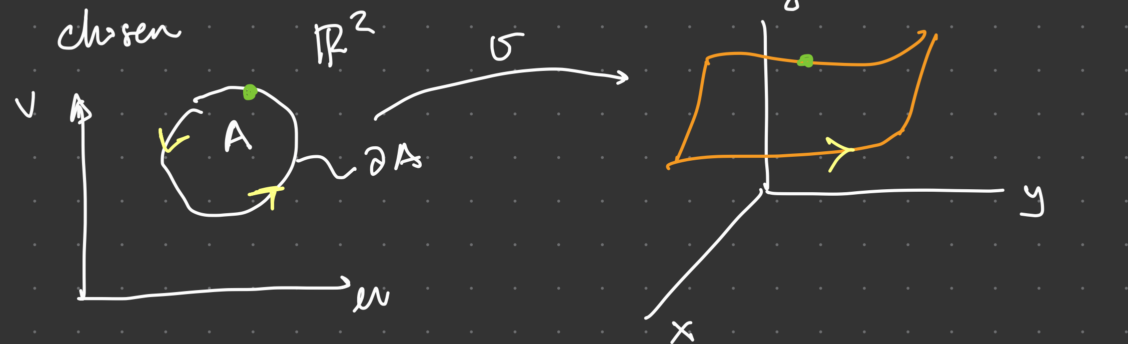
regular domain with boundary ∂B positively oriented and $G : \bar{B} \to \mathbb{R}^2$ s.t. $G \in C^1(B, \mathbb{R}^2)$





3.4.2 Determination of boundary of a surface and its sense of circulation

1) If $\Sigma \subset \mathbb{R}^3$ is a regular surface and $\sigma : \bar{A} \to \Sigma$ then $\partial\Sigma = \sigma(\partial A)$ is independent of the parameterization chosen



2) The sense of circulation of $\partial\Sigma$ is induced by the parameterizat. σ and is obtained by circulating ∂A in the positive sense

3) If Σ is (piecewise) regular surface then
$\sigma(\partial A) = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_m$ and then we process them
as follows:

- we remove from $\sigma(\partial A)$ the curves Γ that reduce to a a point
- we remove curves that are circulated twice
- what remains is the boundary of Σ.