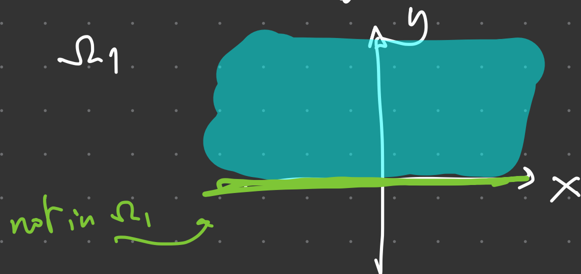
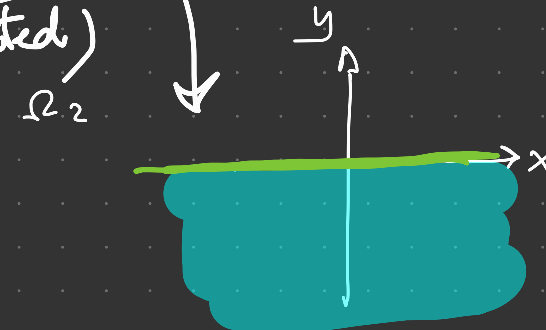
25/09/2025

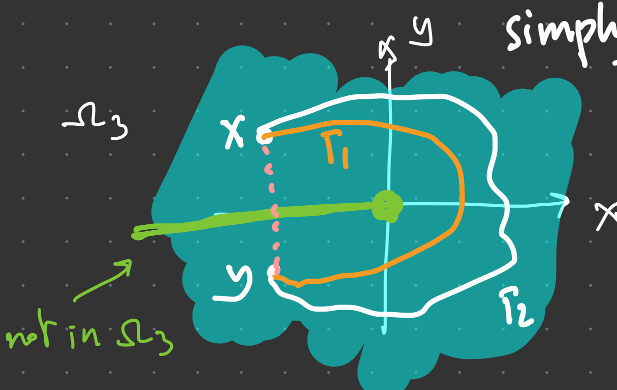
Note: If a domain is convex $\Rightarrow$ it is also simply connected

convex (and simply connected)





simply conn. but not convex



non-convex and non-simply connected

$\Omega_1)$ $\quad F$ is conservative $\Rightarrow$ let's find it!

$f \in C^2(\Omega_1)$ s.t. $\operatorname{grad} f = F$, $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (F_1, F_2)$

$$\frac{\partial f}{\partial x} = F_1(x,y) = \frac{-y}{x^2+y^2} \quad , \quad \int \frac{\partial f}{\partial x}\,dx = \int \frac{-y}{x^2+y^2}\,dx$$

$$f(x,y) = \int \frac{-y}{x^2+y^2}\,dx = -\operatorname{arctg}\left(\frac{x}{y}\right) + \alpha(y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(-\operatorname{arctg}\left(\frac{x}{y}\right) + \alpha(y)\right) = \cancel{\frac{x}{x^2+y^2}} + \alpha'(y) = F_2(x,y) = \cancel{\frac{x}{x^2+y^2}}$$

$$\alpha'(y) = 0 \longrightarrow \alpha(y) = C_1 \quad , \quad C_1 \in \mathbb{R}$$

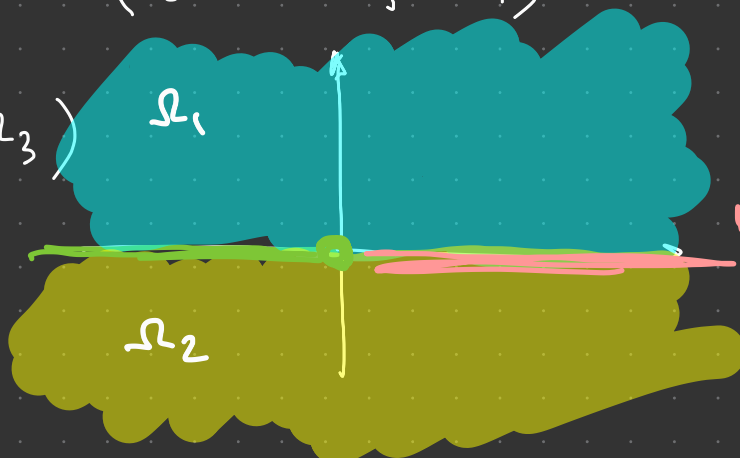
$$f_1(x,y) = -\operatorname{arctg}\left(\frac{x}{y}\right) + c_1, \quad c_1 \in \mathbb{R}$$

for Ω_1

Ω_2) $$f_2(x,y) = -\operatorname{arctg}\left(\frac{x}{y}\right) + c_2, \quad c_2 \in \mathbb{R}.$$

(same as for Ω_1)

Ω_3)



belongs to Ω_3, but doesn't belong to Ω_1 or Ω_2

$$f_3(x,y) = \begin{cases} -\operatorname{arctg}\left(\frac{x}{y}\right) + c_1 & \text{if } y > 0 \quad (\text{for } \Omega_1) \\ -\operatorname{arctg}\left(\frac{x}{y}\right) + c_2 & \text{if } y < 0 \quad (\text{for } \Omega_2) \\ ?? & \text{if } y = 0 \text{ and } x > 0 \end{cases}$$

For $y=0$ and $x>0$

$$\lim_{\substack{y \to 0^+ \\ x>0}} f_3(x,y) = -\lim_{\substack{y\to 0^+ \\ x>0}} \left[\operatorname{arctg}\left(\overbrace{\frac{x}{y}}^{+\infty}\right)\right] + c_1 = -\frac{\pi}{2} + c_1$$

$$\lim_{\substack{y \to 0^- \\ x>0}} f_3(x,y) = -\lim_{\substack{y\to 0^- \\ x>0}} \left[\operatorname{arctg}\left(\underbrace{\frac{x}{y}}_{-\infty}\right)\right] + c_2 = \frac{\pi}{2} + c_2$$

$$-\frac{\pi}{2} + c_1 = \frac{\pi}{2} + c_2 \quad \Rightarrow \quad c_1 = \pi + c_2$$

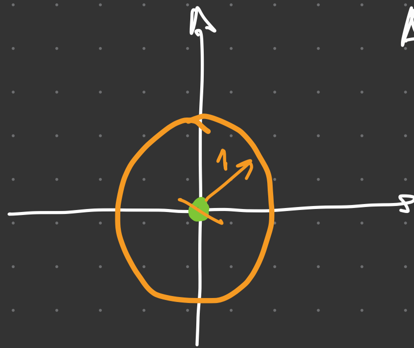
$$f_3(x,y) = \begin{cases} -\operatorname{arctg}\left(\frac{x}{y}\right) + \pi + c_2 & \text{if } y > 0 \\ -\operatorname{arctg}\left(\frac{x}{y}\right) + c_2 & \text{if } y < 0 \\ \frac{\pi}{2} + c_2 & \text{if } y = 0 \text{ and } x > 0 \end{cases}$$

Ω_4) curl $F = 0$, but Ω_4 is non-convex and non-simply connected

$\Rightarrow$ Theorem 1 says nothing.

Let's try Theorem 2: if we find a closed curve $\Gamma \subset \Omega_4$

s.t. $\int_\Gamma F \cdot dl \neq 0$, then F does not derive from a potential.



Ansatz: $\Gamma = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2=1 \}$

$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$

$t \longmapsto \gamma(t) = (\cos t, \sin t)$

$\gamma'(t) = (-\sin t, \cos t)$

$$\int_\Gamma F \cdot dl = \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = 2\pi$$

$\int_\Gamma F \cdot dl \neq 0. \Longrightarrow$ (Theorem 2) F does not derive from a potential in Ω_4.