Example 3: compute the length of Γ, Γ is the circle of radius R and centered at the origin. $\text{length}(\Gamma) = 2\pi R$

$$\text{length}(\Gamma) = \int_\Gamma 1 \, dl$$

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$$
$$t \longmapsto \gamma(t) = R(\cos t, \sin t)$$

$$\text{length}(\Gamma) = \int_\Gamma 1 \cdot dl = \int_0^{2\pi} 1 \; \|\gamma'(t)\| \, dt$$

$\gamma'(t) = R(-\sin t, \cos t)$ " $\|\gamma'(t)\| = R$

$f(x, y) = 1$ " $f(\gamma(t)) = 1$

$$\text{length}(\Gamma) = \int_0^{2\pi} 1 \cdot R = 2\pi R$$

2.3 Fields that derive from a potential

18/9/2025

2.3.1 Description of conservative f.

• **Definition** : let $\Omega \subset \mathbb{R}^n$ be an open domain and

$$F : \Omega \to \mathbb{R}^n$$
$$x \mapsto F(x) = (F_1(x), ..., F_n(x))$$ a vector field

We say that F derives from a potential in Ω if there exists a scalar field $f : \Omega \to \mathbb{R}$, $x \mapsto f(x)$ of class $f \in C^1(\Omega)$

s.t. $F = \operatorname{grad} f = \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right)$

In this case, F is a conservative field and f is the potential

Remark:

1) If the potential exists, then it is defined up to constant $\alpha \in \mathbb{R}$

$$F = \operatorname{grad}(f) = \operatorname{grad}(f + \alpha)$$

2) classical example:

$$f = \frac{c}{r} + \alpha \quad , \quad r = \text{dist.} \quad , c = g\,m\,M$$

$$F = \frac{-c}{r^3}(x, y, z) \quad , \quad F = \operatorname{grad} f .$$

it is the potential.

vector field is a conservative field

2.3.2 Important results

- Theorem 1: let $\Omega \subset \mathbb{R}^n$ be an open domain and

$$F : \Omega \to \mathbb{R}^n, \quad x \mapsto F(x)$$

a vector field s.t. $F \in C^1(\Omega, \mathbb{R}^n)$

a) Necessary condition: If F derives from a potential in Ω

then $$\frac{\partial F_i}{\partial x_j}(x) = \frac{\partial F_j}{\partial x_i}(x), \quad \forall\, i,j = 1, 2, \dots, n \quad (*)$$

and $\forall\, x \in \Omega$

b) sufficient condition:

If $(*)$ holds and if Ω is convex and/or simply connected, then F derives from a potential in Ω.

Remarks:

1) The condition (*) is necessary but not sufficient

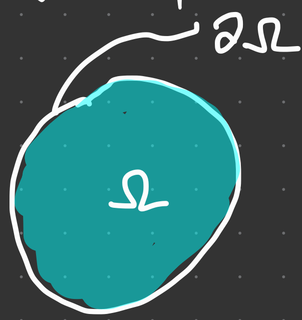
2) The condition (*) is equivalent to curl $F = 0$

E.g. $n = 2$ curl $F = \dfrac{\partial F_2}{\partial x_1} - \dfrac{\partial F_1}{\partial x_2} = 0$

The same for $n=3$.

Recalls (or definitions)

1)



$\Omega \subset \mathbb{R}^n$ is open if $\partial\Omega$ doesn't belong to Ω.

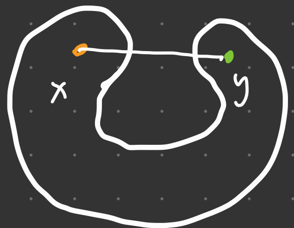
2) $\Omega \subset \mathbb{R}^n$ is connected if $\forall x, y \in \Omega$ $\exists$ a curve Γ that joins x and y that is fully contained in Ω

connected

non-connected

3) $\Omega \subset \mathbb{R}^n$ is convex if $\forall t \in [0,1]$ and $x, y \in \Omega$ we have $x + t(y-x) \in \Omega$ (i.e., the segment between x and y) is fully contained in Ω.

convex



non-convex
but connected

non-convex
and non connected

if Ω is non connected $\Longrightarrow$ is non convex

4) $\Omega \subset \mathbb{R}^n$ is simply connected if $\forall x, y \in \Omega$
I consider two arbitrary curves Γ_1 and Γ_2 that connect x and y, then Γ_1 can be deformed into Γ_2 without getting out of Ω.