EPFL - Fall 2025	Dr. Pablo Antolin
Analysis III – MATH-202(c) – EL GM MX	**Exercises**
Series 14	December 16 2025

Exercise 1.

1. Let $w, \xi > 0$, and let $f, \varphi\colon \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$f(x) = x^2 e^{-w^2x^2}, \qquad \varphi(x) = \frac{2w^2 + \xi^2 - 4w^4x^2}{4w^4}\, e^{-w^2x^2}.$$

   Show (using Fourier transform tables and standard Fourier transform properties) that

$$\hat{f}(\alpha) = \frac{2w^2 - \alpha^2}{4\sqrt{2}\,w^5}\, e^{-\frac{\alpha^2}{4w^2}}, \qquad \hat{\varphi}(\alpha) = \frac{\xi^2 + \alpha^2}{4\sqrt{2}\,w^5}\, e^{-\frac{\alpha^2}{4w^2}}.$$

2. Find a function $u\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$u(x) + \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} u''(t)\, e^{-\sqrt{2}|x-t|}\, dt = x^2 e^{-x^2}.$$

Exercise 2.
We check whether the following are distributions or not.

1. Show that T is a distribution, where

$$T(\phi) = \int_{-1}^{1} \phi(x) dx. \tag{1}$$

2. Show that T is a distribution, where

$$T(\phi) = \int_{-\infty}^{\infty} \phi(x) dx. \tag{2}$$

3. Show that S is **not** a distribution, where

$$S(\phi) = \int_{0}^{1} |\phi(x)| dx. \tag{3}$$

Exercise 3.
Find a function $u\colon \mathbb{R} \to \mathbb{R}$, 2π-periodic, satisfying

$$u''(x) - 2u(x+\pi) = 3 + \sin\left(\tfrac{3}{2}x\right) \sin\left(\tfrac{x}{2}\right).$$

Hint: You may use the identities

$$2\sin(a)\sin(b) = \cos(a-b) - \cos(a+b),$$

$$\sin\big(n(x\pm\pi)\big) = (-1)^n \sin(nx), \qquad \cos\big(n(x\pm\pi)\big) = (-1)^n \cos(nx).$$

Exercise 4.
Find the distributional derivative of the function

$$f(x) = |x|. \tag{4}$$

Exercise 5.
Solve for $u = u(x,t)$

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & x \in (0,\pi), t > 0 \\ u(0,t) = u(\pi,t) = 0 & t > 0 \\ u(x,0) = \cos x - \cos(3x) & x \in (0,\pi) \end{cases}$$

Exercise 6.
Find $u = u(x,y)$ satisfying

$$\begin{cases} \Delta u = 0 & x, y \in (0,\pi), \\ \dfrac{\partial u}{\partial y}(x,0) = \dfrac{\partial u}{\partial y}(x,\pi) = 0 & x \in (0,\pi), \\ u(0,y) = \cos(2y), \quad u(\pi,y) = 0 & y \in (0,\pi). \end{cases}$$