

*Hint* : In order to verify Green's theorem (e.g., exercises 1 and 2), apply the following steps:

1. Try to sketch the domain  $\Omega$  and its boundary  $\partial\Omega$ . Indicate the direction of the curve  $\partial\Omega$  so that it is positively oriented.
2. Compute  $\text{curl}F(x, y)$ .
3. Parameterize the domain  $\Omega$ , and use this parameterization to compute

$$\iint_{\Omega} \text{curl}F \, dx dy.$$

4. Parametrize the boundary  $\partial\Omega$  of  $\Omega$ , and use this parameterization to compute

$$\int_{\partial\Omega} F \cdot dl.$$

5. To verify the Green's theorem, check that both integrals are equal.

**Exercise 1** (Ex 4.1 and Ex 4.2 page 41).

Verify Green's theorem in the following cases:

1.  $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$  and  $F(x, y) = (xy, y^2)$ .
2.  $A = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$  and  $F(x, y) = (x + y, y^2)$ .

**Exercise 2** (Ex 4.4i and Ex 4.5 page 42).

Verify Green's theorem in the following cases:

1.  $A = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - 1)^2 < 1\}$  and  $F(x, y) = (-x^2y, xy^2)$ .
2.  $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1 \text{ and } x^2 - 4 < y < 2\}$  and  $F(x, y) = (xy, y)$ .

*Hint*: It will be difficult to parametrize the domain with a single parametrization. Try to sketch the domain  $A$  and then, you have two options:

- Compute  $\iint_A \text{rot} F(x, y) \, dx dy$  as one integral minus another. (Recommended since it is faster)
- Split the domain into four  $x$ -simple or  $y$ -simple parts. (Not recommended since it is longer)

**Exercise 3** (Ex 4.3 page 42).

Let  $\Omega \subset \mathbb{R}^2$  be a triangle whose vertices are  $(0,0)$ ,  $(0,1)$ , and  $(1,0)$ . Let  $f(x,y) = y + e^x$ . Compute:

1.  $\int_{\Omega} \Delta f(x,y) \, dx dy$ .
2.  $\int_{\partial\Omega} \left( \frac{\partial f}{\partial x} \nu_1 + \frac{\partial f}{\partial y} \nu_2 \right) dl$ , where  $\nu = (\nu_1, \nu_2)$  is the outer unit vector normal of the boundary  $\partial\Omega$ .

**Exercise 4** (Ex 4.9i page 43).

Let  $\Omega \subset \mathbb{R}^2$  be a regular domain which has a positively oriented boundary  $\partial\Omega$ . Let  $F$ ,  $G_1$ , and  $G_2$  be the vector fields defined as

$$F(x,y) = (-y, x), \quad G_1(x,y) = (0, x) \quad \text{and} \quad G_2(x,y) = (-y, 0).$$

Prove that:

1.  $\text{Area}(\Omega) = \frac{1}{2} \int_{\partial\Omega} F \cdot dl$ .
2.  $\text{Area}(\Omega) = \int_{\partial\Omega} G_1 \cdot dl$ .
3.  $\text{Area}(\Omega) = \int_{\partial\Omega} G_2 \cdot dl$ .