EPFL - Fall 2025	Dr. Pablo Antolin
Analysis III – MATH-202(c) – EL GM MX	**Exercises**
Series 1	September 9 2025

Remark.
The theorems introduced in this Series were studied in Analysis II.

Definition 1.
Let $\Omega \subset \mathbb{R}^m$ be an open set, $\boldsymbol{F} = (F_1, \ldots, F_n) : \Omega \to \mathbb{R}^n$, such that $\boldsymbol{F} \in C^1(\Omega, \mathbb{R}^n)$ and $1 \leq i \leq m$. We define:

$$\frac{\partial \boldsymbol{F}}{\partial x_i} = \left(\frac{\partial F_1}{\partial x_i}, \ldots, \frac{\partial F_n}{\partial x_i} \right).$$

Exercise 1.
Let $\Omega \subset \mathbb{R}^m$ be an open set, $\boldsymbol{F}, \boldsymbol{G} \in C^1(\Omega, \mathbb{R}^n)$ and $1 \leq i \leq m$. Show that:

1. $$\frac{\partial}{\partial x_i}[\langle \boldsymbol{F}, \boldsymbol{G} \rangle] = \left\langle \frac{\partial \boldsymbol{F}}{\partial x_i}, \boldsymbol{G} \right\rangle + \left\langle \boldsymbol{F}, \frac{\partial \boldsymbol{G}}{\partial x_i} \right\rangle,$$

   where for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ denotes the dot product of these two Euclidean vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and is defined by:

   $$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \sum_{i=1}^{n} a_i b_i,$$

2. In three-dimensional space ($n = 3$),

   $$\frac{\partial}{\partial x_i}[\boldsymbol{F} \wedge \boldsymbol{G}] = \frac{\partial \boldsymbol{F}}{\partial x_i} \wedge \boldsymbol{G} + \boldsymbol{F} \wedge \frac{\partial \boldsymbol{G}}{\partial x_i},$$

   where for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, $\boldsymbol{a} \wedge \boldsymbol{b} \in \mathbb{R}^3$ denotes the cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$, and is defined by:

   $$\boldsymbol{a} \wedge \boldsymbol{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.$$

Definition 2 (Jacobian matrix and determinant)**.**
Let $\boldsymbol{u} = (u_1, \ldots, u_n) : \Omega \subset \mathbb{R}^n \to \Omega' \subset \mathbb{R}^n, \boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ such that:

- $\boldsymbol{u} \in C^\infty(\Omega; \Omega')$,

- $\boldsymbol{u}$ is invertible and $\boldsymbol{u}^{-1} \in C^{\infty}(\Omega'; \Omega)$.

The Jacobian matrix of the vector-valued function $\boldsymbol{u}$, represented by $\nabla \boldsymbol{u}$, is a square matrix whose entries are:

$$(\nabla \boldsymbol{u})_{i,j} = \frac{\partial \boldsymbol{u}_i}{\partial x_j}.$$

The Jacobian determinant, or simply the Jacobian, denoted by $\operatorname{Jac} \boldsymbol{u}(\boldsymbol{x})$ is defined as:

$$\operatorname{Jac} \boldsymbol{u}(\boldsymbol{x}) = \det \nabla \boldsymbol{u}(\boldsymbol{x}).$$

Exercise 2 (§8.5 pages 119-121)**.**
Compute the Jacobian for the following mapping functions:

1. Polar coordinates:
$$\boldsymbol{u}(r, \theta) = (r \cos \theta, r \sin \theta),$$
2. Spherical coordinates:
$$\boldsymbol{u}(r, \theta, \varphi) = (r \cos \theta \sin \varphi, r \sin \theta \sin \varphi, r \cos \varphi),$$
3. Cylindrical coordinates:
$$\boldsymbol{u}(r, \theta, z) = (r \cos \theta, r \sin \theta, z),$$
4. Cartesian coordinates:
$$\boldsymbol{u}(x, y, z) = (x, y, z).$$

Theorem 3 (§8.5 page 119)**.**
Let $f : \Omega' \to \mathbb{R}$ be an continuous function, $\boldsymbol{u} : \Omega \to \Omega'$ be a function as the one involved in Definition 2, and $A \subset \Omega'$ be a closed and bounded set. Then:

$$\int_A f(\boldsymbol{x}) d\boldsymbol{x} = \int_{\boldsymbol{u}^{-1}(A)} f(\boldsymbol{u}(\tilde{\boldsymbol{x}})) \, |\operatorname{Jac} \boldsymbol{u}(\tilde{\boldsymbol{x}})| \, d\tilde{\boldsymbol{x}}.$$

Exercise 3.
Sketch the set A, and compute the integral $\int_A f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$ in the following cases:

1. $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 4\}$, and $f(x, y) = (x^2 + y^2)^{-\frac{1}{2}}$,
2. $A = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 3 \, , \, x - 3 \le y \le 3 - x\}$,
   and $f(x, y) = x^2 + \sin^3(y)$,
3. $A = \{(x, y) \in \mathbb{R}^2 \mid -1 \le x \le 1 \, , \, 0 \le y \le \sqrt{1 - x^2}\}$,
   and $f(x, y) = y\,(1 + x^2)$,
4. $A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 9\}$, and $f(x, y, z) = \sqrt{x^2 + y^2}$,
5. $A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 4\}$,
   and $f(x, y, z) = \arccos \frac{z}{\sqrt{x^2+y^2+z^2}}$.

Theorem 4.
Let $\Omega \subset \mathbb{R}^n, \Omega' \subset \mathbb{R}^m, f \in C^1(\Omega')$ and $g = (g_1, \ldots, g_m) \in C^1(\Omega, \mathbb{R}^m)$ such that $g(\Omega) \subset \Omega'$. Then, the chain rule expresses the derivatives of the composition function $f \circ g \in C^1(\Omega)$ as:

$$\frac{\partial f \circ g}{\partial x_i}(x) = \sum_{j=1}^{m} \frac{\partial f}{\partial x_j}(g(x)) \frac{\partial g_j}{\partial x_i}(x), \quad \text{for } i = 1 \ldots n.$$

Exercise 4.

1. Let $p \ge 1$ and the function:

   $$h_p(x) := |x|^p = \left(\sqrt{x_1^2 + \cdots + x_n^2}\right)^p, \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n \backslash \{0\}.$$

   Compute $\nabla h_p(x)$.
2. Let $\boldsymbol{G} \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ be a map that never reaches zero, and f_p the function defined by:

   $$f_p(t) := \frac{1}{p} |\boldsymbol{G}(tx)|^p, \quad \forall t \in \mathbb{R}.$$

   Compute $\frac{\mathrm{d}}{\mathrm{d}t} f_p(t)$.