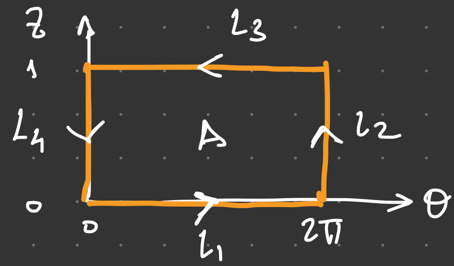
Week 7 – part 1

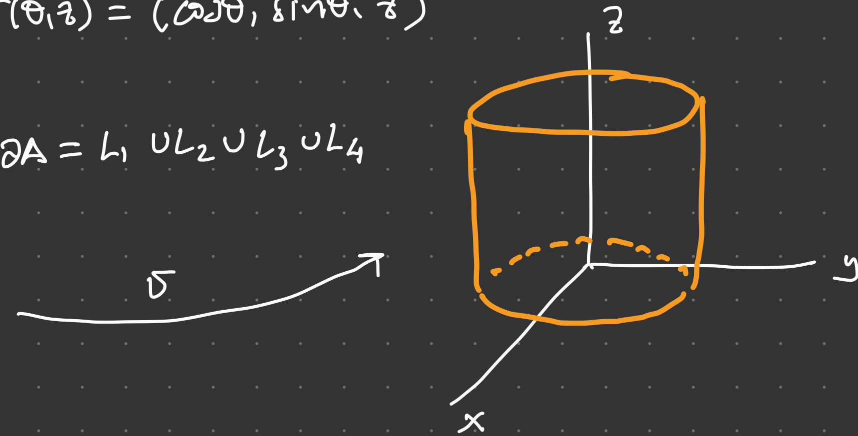
Example 1: cylinder $\Sigma = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2=1 , \ 0<z<1 \}$

$$A =]0,2\pi[\times]0,1[$$

$$\sigma : \overline{A} \longrightarrow \mathbb{R}^3$$
$$(\theta, z) \longmapsto \sigma(\theta,z) = (\cos\theta, \sin\theta, z)$$



$$\partial A = L_1 \cup L_2 \cup L_3 \cup L_4$$



$$\partial \Sigma = \sigma(\partial A) = \sigma(L_1 \cup L_2 \cup L_3 \cup L_4)$$
$$= \sigma(L_1) \cup \sigma(L_2) \cup \sigma(L_3) \cup \sigma(L_4)$$

$$L_1 = \{ \alpha_1(\theta) = (\theta, 0) \quad \text{with } \theta : 0 \to 2\pi \}$$

$$L_2 = \{ \alpha_2(z) = (2\pi, z) \text{ with } z : 0 \to 1 \}$$

$$L_3 = \{ \alpha_3(\theta) = (\theta, 1) \quad \text{with } \theta : 2\pi \to 0 \}$$

$$L_4 = \{ \alpha_4(z) = (0, z) \quad \text{with } z : 1 \to 0 \}$$

$$\Gamma_1 = \sigma(L_1(\theta)) = \sigma \circ L_1(\theta)$$

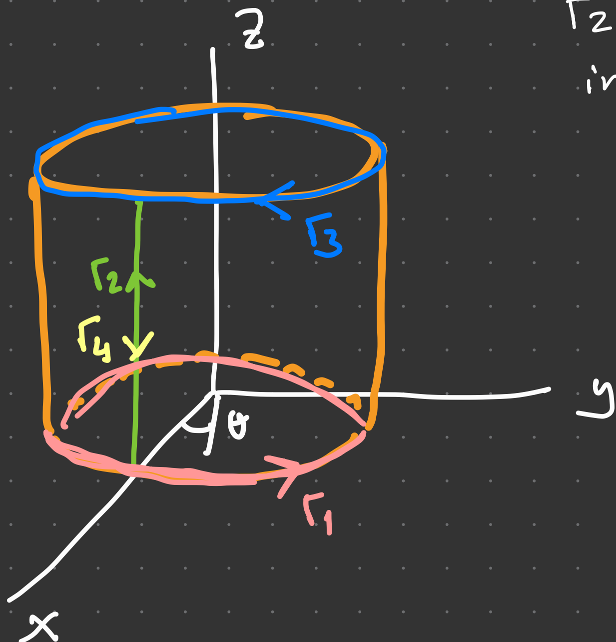
$$\sigma(\theta, z) = (\cos\theta, \sin\theta, z)$$

$$\Gamma_1 = \{ \gamma_1(\theta) = \sigma(\theta, 0) = (\cos\theta, \sin\theta, 0) \text{ with } \theta : 0 \to 2\pi \}$$

$$\Gamma_2 = \{ \gamma_2(z) = \sigma(2\pi, z) = (1, 0, z) \text{ with } z : 0 \to 1 \}$$

$$\Gamma_3 = \{ \gamma_3(\theta) = \sigma(\theta, 1) = (\cos\theta, \sin\theta, 1) \text{ with } \theta : 2\pi \to 0 \}$$

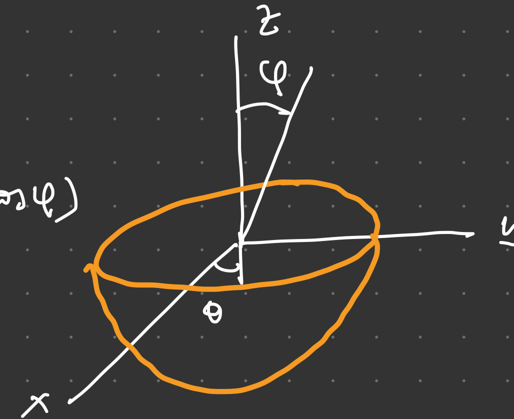
$$\Gamma_4 = \{ \gamma_4(z) = \sigma(0, z) = (1, 0, z) \text{ with } z : 1 \to 0 \}$$



Γ_2 and Γ_4 are the same, but circulated in opposite sense $\longrightarrow$ I remove them both.

$$\partial\Sigma = \Gamma_1 \cup \Gamma_3$$

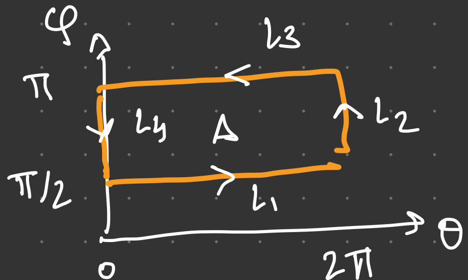
Example 2: semi-sphere : $\{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1 \text{ and } z \leq 0 \}$



$A =]0, 2\pi[\;]\frac{\pi}{2}, \pi[$

$\sigma : \bar{A} \longrightarrow \mathbb{R}^3$

$(\theta, \varphi) \longmapsto \sigma(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$



$\partial\Sigma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$

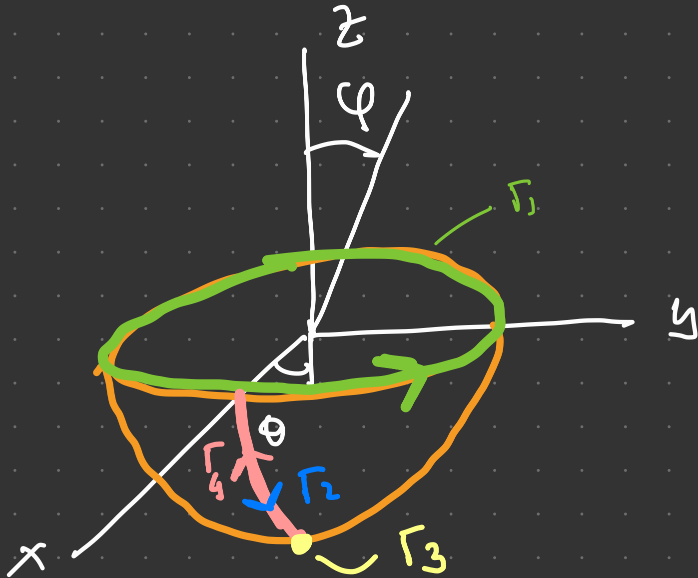
$\Gamma_1 = \{ \gamma_1(\theta) = \sigma(\theta, \pi/2) = (\cos\theta, \sin\theta, 0)$ with $\theta : 0 \longrightarrow 2\pi \}$

$\Gamma_2 = \{ \gamma_2(\varphi) = \sigma(2\pi, \varphi) = (\sin\varphi, 0, \cos\varphi)$ with $\varphi : \pi/2 \rightarrow \pi \}$

$\Gamma_3 = \{ \gamma_3(\theta) = \sigma(\theta, \pi) = (0, 0, -1)$ with $\theta : 2\pi \longrightarrow 0 \}$ → a point

$\Gamma_4 = \{ \gamma_4(\varphi) = \sigma(0, \varphi) = (\sin\varphi, 0, \cos\varphi)$ with $\varphi : \pi \rightarrow \pi/2 \}$

$\partial\Sigma = \Gamma_1$

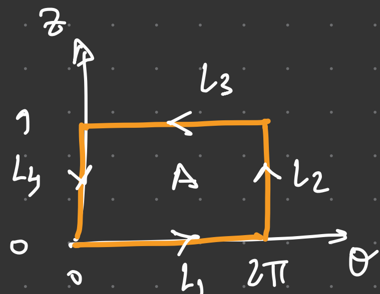


Example 3: cone $= \{ (x,y,z) \in \mathbb{R}^3 : z^2 = x^2+y^2$ and $0 < z < 1 \}$

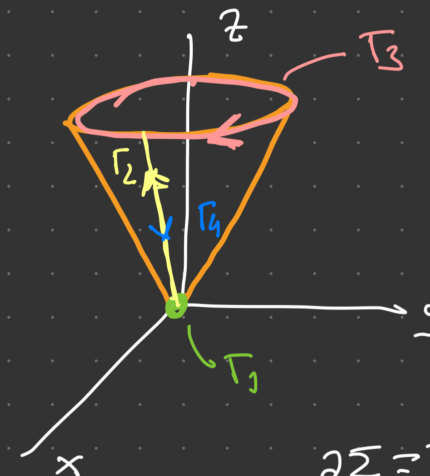
$A = \,]0, 2\pi[\times]0,1[$

$\sigma : \bar{A} \longrightarrow \mathbb{R}^3$

$(\theta, z) \longmapsto \sigma(\theta, z) = (z\cos\theta, z\sin\theta, z)$



$\partial\Sigma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$



$\partial\Sigma = \Gamma_3$

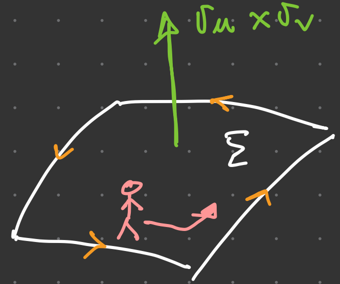
$\Gamma_1 = \{ \gamma_1(\theta) = \sigma(\theta, 0) = (0,0,0)$ with $\theta : 0 \to 2\pi \}$

$\Gamma_2 = \{ \gamma_2(z) = \sigma(2\pi, z) = (z, 0, z)$ with $z : 0 \to 1 \}$

$\Gamma_3 = \{ \gamma_3(\theta) = \sigma(\theta, 1) = (\cos\theta, \sin\theta, 1)$ with $\theta : 2\pi \to 0 \}$

$\Gamma_4 = \{ \gamma_4(z) = \sigma(0, z) = (z, 0, z)$ with $z : 1 \to 0 \}$

Note: practical rule for obtaining the boundary of a surface induced by a parameterization $\sigma(u,v)$: an observer walks along $\partial\Sigma$, with his/her head pointing in the sense of the normal $\sigma_u \times \sigma_v$, the surface Σ is on his/her left.



Week 7 - part 2

3.4.3 Stokes' theorem

- Theorem: let $\Sigma \subset \mathbb{R}^3$ be a (piecewise) orientable regular surface.
  let $F: \Sigma \to \mathbb{R}^3$ be a vector field s.t. $F \in C^1(\bar{\Sigma}, \mathbb{R}^3)$ then:

$$\iint_{\Sigma} \operatorname{curl} F \cdot ds = \int_{\partial\Sigma} F \cdot dl$$

- Remarks:

  1- This is a generalization of Green's theorem for $\mathbb{R}^3$.

  2- Once the parameterization $\sigma : \bar{A} \to \Sigma$, $(u,v) \to \sigma(u,v)$ is chosen

  we take $\sigma_u \times \sigma_v$ as the normal vector in the surface integral.

  I.e. $\iint_{\Sigma} \operatorname{curl} F \cdot ds = \iint_{A} \operatorname{curl} F(\sigma(u,v)) \cdot \sigma_u \times \sigma_v \, du\, dv$

3 - The sense of circulation of $\partial\Sigma$ in the integral $\int_{\partial\Sigma} F \cdot dl$ is induced by σ. I.e, it is obtained by circulating ∂A in positive sense (counter-clockwise).

week 7 - part 3

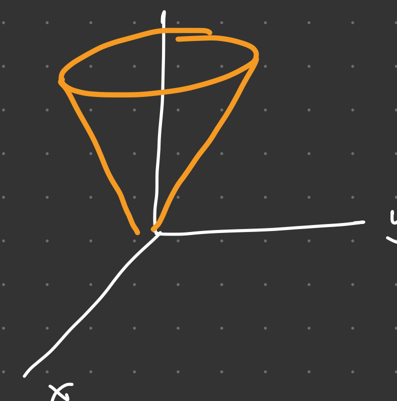
3.4.4 Stokes' theorem examples

- Example 1 : Verify the Stokes' theorem for

$\Sigma = \{ (x,y,z) \in \mathbb{R}^3 : z^2 = x^2 + y^2$ and $0 < z < 1 \}$

and $F : \Sigma \to \mathbb{R}^3$

$(x,y,z) \mapsto F(x,y,z) = (z, x, y)$



$A =]0, 2\pi[\times]0, 1[$

$\sigma : \bar{A} \to \mathbb{R}^3$

$(\theta, z) \mapsto \sigma(\theta, z) = (z\cos\theta, z\sin\theta, z)$

- $\iint_\Sigma \operatorname{curl} F \cdot ds$ " $\quad \operatorname{curl} F = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \sigma_\theta \times \sigma_z = \begin{pmatrix} z\cos\theta \\ z\sin\theta \\ -z \end{pmatrix}$

$$\overset{\shortparallel}{\iint_A} (1,1,1) \cdot (z\cos\theta, z\sin\theta, -z)\, dz\, d\theta = \int_0^{2\pi}\int_0^1 z(\cos\theta + \sin\theta - 1)\, dz\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (\underbrace{\cos\theta + \sin\theta}_{=0} - 1)\, d\theta = -\frac{1}{2}\int_0^{2\pi} d\theta = -\pi$$

- $\int_{\partial\Sigma} F \cdot dl \quad , \quad \Gamma = \partial\Sigma, \quad \gamma : [2\pi, 0] \to \mathbb{R}^3$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \theta \mapsto \gamma(\theta) = (\cos\theta, \sin\theta, 1)$

$$\overset{\shortparallel}{\int_{2\pi}^{0}} F(\gamma(\theta)) \cdot \gamma'(\theta)\, d\theta = \int_{2\pi}^{0} (1, \cos\theta, \sin\theta) \cdot (-\sin\theta, \cos\theta, 0)\, d\theta$$

$$\left(\int_0^{2\pi} \cos^2\theta\, d\theta = \pi \right)$$

$$= \int_{2\pi}^{0} (-\sin\theta + \cos^2\theta)\, d\theta = \int_{2\pi}^{0} \cos^2\theta\, d\theta = -\pi$$

Homework:

$$\sigma(z,\theta) = (z\cos\theta,\ z\sin\theta,\ z),$$

normal: $\sigma_z \times \sigma_\theta = -\sigma_\theta \times \sigma_z = \begin{pmatrix} -z\cos\theta \\ -z\sin\theta \\ +z \end{pmatrix}$

$\partial\Sigma$ has the opposite orientation

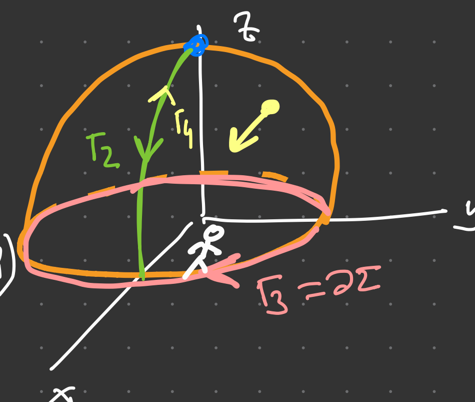
Example 2: $\Sigma = \{(x,y,z) \in \mathbb{R}^3 :\ x^2+y^2+z^2 = 1 \text{ and } z > 0\}$

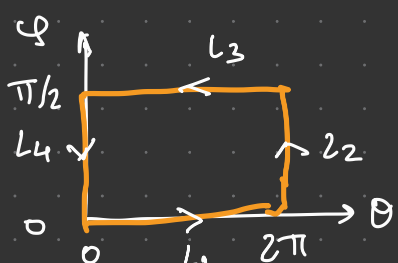
$F(x,y,z) = (z,\ x,\ y^2)$

$A = \,]0, 2\pi[\times]0, \pi/2[$

$\sigma : \bar{A} \longrightarrow \mathbb{R}^3$

$(\theta,\varphi) \longmapsto \sigma(\theta,\varphi) = (\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)$





$\partial\Sigma$?

$\times\ \Gamma_1 = \{ \gamma_1(\theta) = \sigma(\theta, 0) = (0, 0, 1)$ with $\theta : 0 \to 2\pi \}$

$\times\ \Gamma_2 = \{ \gamma_2(\varphi) = \sigma(2\pi, \varphi) = (\sin\varphi, 0, \cos\varphi)$ with $\varphi : 0 \to \pi/2 \}$

$\Gamma_3 = \{ \gamma_3(\theta) = \sigma(\theta, \pi/2) = (\cos\theta, \sin\theta, 0)$ with $\theta : 2\pi \to 0 \}$

$\times\ \Gamma_4 = \{ \gamma_4(\varphi) = \sigma(0, \varphi) = (\sin\varphi, 0, \cos\varphi)$ with $\varphi : \pi/2 \to 0 \}$

- $$\int_{\partial\Sigma} F \cdot dl = \int_{2\pi}^{0} F(\gamma_3(\theta)) \cdot \gamma_3'(\theta)\, d\theta = \int_{2\pi}^{0} (0, \cos\theta, \sin^2\theta) \cdot (-\sin\theta, \cos\theta, 0)\, d\theta$$

$$= \int_{2\pi}^{0} \cos^2\theta\, d\theta = -\pi.$$

- $$\iint_{\Sigma} \operatorname{curl} F \cdot ds = \iint_{A} [(\operatorname{curl} F) \circ \sigma(\theta, \varphi)] \cdot (\sigma_\theta \times \sigma_\varphi)\, d\theta\, d\varphi.$$

$\operatorname{curl} F = (2y, 1, 1)$, $\quad \operatorname{curl} F \circ \sigma(\theta, \varphi) = (2\sin\varphi \sin\theta, 1, 1)$

$$\sigma_\theta \times \sigma_\varphi = -\sin\varphi \begin{pmatrix} \sin\varphi\cos\theta \\ \sin\varphi \sin\theta \\ \cos\varphi \end{pmatrix} = -\sin\varphi\, \sigma(\theta, \varphi)$$

$$\iint_\Sigma \operatorname{curl} F \cdot dS = -\int_0^{\pi/2}\int_0^{2\pi} (2\sin\varphi\sin\theta, 1, 1)\cdot \sigma(\theta,\varphi) \sin\varphi \, d\theta \, d\varphi$$

$\int_0^{2\pi} \sin\theta\cos\theta \, d\theta = 0$

$$= -\int_0^{\pi/2}\int_0^{2\pi} \left(\cancel{2\sin^3\varphi \sin\theta\cos\theta} + \cancel{\sin^2\varphi \sin\theta} + \sin\varphi\cos\varphi\right) d\theta \, d\varphi$$

$$= -2\pi \int_0^{\pi/2} \sin\varphi\cos\varphi \, d\varphi = -2\pi \, \frac{1}{2} \sin^2\varphi \Big|_0^{\pi/2} = -\pi$$