EPFL - Fall 2025	Dr. Pablo Antolin
Analysis III – MATH-202(c) – EL GM MX	**Exercises**
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Exercise 1.

1. Two methods are given for computing $\hat{f}$:

   Method 1 :

$$\begin{aligned}
\hat{f}(\alpha) &= \mathcal{F}\Big[x \cdot \Big(xe^{-w^2x^2}\Big)\Big](\alpha) \\
\text{Prop. 5.7 (ii)} \qquad &= i\frac{d}{d\alpha}\left[\mathcal{F}\Big[xe^{-w^2x^2}\Big](\alpha)\right] \\
\text{Table §15.5, 9)} \qquad &= i\frac{d}{d\alpha}\left[\frac{-i\alpha}{2\sqrt{2}\,w^3}\,e^{-\frac{\alpha^2}{4w^2}}\right] \\
&= i\left(\frac{-i}{2\sqrt{2}\,w^3}e^{-\frac{\alpha^2}{4w^2}} + \frac{-i\alpha}{2\sqrt{2}\,w^3}\left(-\frac{\alpha}{2w^2}\right)e^{-\frac{\alpha^2}{4w^2}}\right) \\
&= \frac{2w^2-\alpha^2}{4\sqrt{2}\,w^5}\,e^{-\frac{\alpha^2}{4w^2}}.
\end{aligned}$$

   Method 2 :

   Observe that

$$\frac{d}{dx}\left[xe^{-w^2x^2}\right] = e^{-w^2x^2} - 2w^2x^2e^{-w^2x^2} = e^{-w^2x^2} - 2w^2f(x),$$

   and therefore

$$f(x) = \frac{1}{2w^2}\left(e^{-w^2x^2} - \frac{d}{dx}\left[xe^{-w^2x^2}\right]\right).$$

   One may also obtain this using integration by parts:

$$\int^x f(t)\,dt = \int^x t^2e^{-w^2t^2}\,dt \overset{\text{IBP}}{=} -\frac{x}{2w^2}e^{-w^2x^2} + \int^x \frac{1}{2w^2}e^{-w^2t^2}\,dt \qquad \left|\begin{array}{ll} u=t & v=-\frac{1}{2w^2}e^{-w^2t^2} \\ u'=1 & v'=te^{-w^2t^2}\end{array}\right.$$

   and hence

$$f(x) = \frac{d}{dx}\left[\int^x f(t)\,dt\right] = -\frac{1}{2w^2}\frac{d}{dx}\left[xe^{-w^2x^2}\right] + \frac{1}{2w^2}e^{-w^2x^2} = \frac{1}{2w^2}\left(e^{-w^2x^2} - \frac{d}{dx}\left[xe^{-w^2x^2}\right]\right).$$

Thus,

$$
\begin{aligned}
\hat{f}(\alpha) &= \frac{1}{2w^2}\left(\mathcal{F}\left[e^{-w^2x^2}\right](\alpha) - \mathcal{F}\left[\frac{d}{dx}\left[xe^{-w^2x^2}\right]\right](\alpha)\right) \\
\text{Prop. 5.7 (i)} \qquad &= \frac{1}{2w^2}\left(\mathcal{F}\left[e^{-w^2x^2}\right](\alpha) - (i\alpha)\mathcal{F}\left[xe^{-w^2x^2}\right](\alpha)\right) \\
\text{Table §15.5, 8) \& 9)} \qquad &= \frac{1}{2w^2}\left(\frac{1}{\sqrt{2}w}e^{-\frac{\alpha^2}{4w^2}} - (i\alpha)\frac{-i\alpha}{2\sqrt{2}\,w^3}e^{-\frac{\alpha^2}{4w^2}}\right) \\
&= \frac{2w^2-\alpha^2}{4\sqrt{2}\,w^5}\,e^{-\frac{\alpha^2}{4w^2}}.
\end{aligned}
$$

Now let us compute $\hat{\varphi}$. We have

$$
\varphi(x) = \frac{2w^2+\xi^2}{4w^4}e^{-w^2x^2} - f(x),
$$

hence

$$
\begin{aligned}
\hat{\varphi}(\alpha) &= \frac{2w^2+\xi^2}{4w^4}\mathcal{F}\left[e^{-w^2x^2}\right](\alpha) - \hat{f}(\alpha) \\
\text{Table §15.5, 8)} \qquad &= \frac{2w^2+\xi^2}{4\sqrt{2}\,w^5}e^{-\frac{\alpha^2}{4w^2}} - \frac{2w^2-\alpha^2}{4\sqrt{2}\,w^5}e^{-\frac{\alpha^2}{4w^2}} \\
&= \frac{\xi^2+\alpha^2}{4\sqrt{2}\,w^5}e^{-\frac{\alpha^2}{4w^2}},
\end{aligned}
$$

which is the desired result.

2. Let $h(x) = \frac{1}{\sqrt{2}}e^{-\sqrt{2}|x|}$. Then Table §15.5, item 7), gives

$$
\hat{h}(\alpha) = \sqrt{\frac{2}{\pi}}\frac{1}{2+\alpha^2}.
$$

The equation can thus be written (take $w = 1$ for the function f from (i))

$$
u(x) + (u'' * h)(x) = f(x).
$$

Applying the Fourier transform to the equation and using Propositions 5.7 (i) (derivative transform), 5.12 (convolution transform), together with the result of (i), we obtain

$$\begin{aligned}
\hat{u}(\alpha) + \sqrt{2\pi}\,\mathcal{F}[u''](\alpha)\,\hat{h}(\alpha) &= \hat{f}(\alpha) \\
\hat{u}(\alpha) + \sqrt{2\pi}(-\alpha^2)\hat{u}(\alpha)\sqrt{\frac{2}{\pi}}\frac{1}{2+\alpha^2} &= \frac{2-\alpha^2}{4\sqrt{2}}\,e^{-\frac{\alpha^2}{4}} \\
\hat{u}(\alpha)\left(1 - \frac{2\alpha^2}{2+\alpha^2}\right) &= \frac{2-\alpha^2}{4\sqrt{2}}e^{-\frac{\alpha^2}{4}} \\
\hat{u}(\alpha)\frac{2-\alpha^2}{2+\alpha^2} &= \frac{2-\alpha^2}{4\sqrt{2}}e^{-\frac{\alpha^2}{4}} \\
\hat{u}(\alpha) &= \frac{2+\alpha^2}{4\sqrt{2}}e^{-\frac{\alpha^2}{4}}.
\end{aligned}$$

Using the result of (i) with $w = 1$ and $\xi = \sqrt{2}$, we conclude that

$$u(x) = \varphi(x) = (1 - x^2)e^{-x^2}.$$

Exercise 2.

The definition in (1) is clearly linear in ϕ as the integral is linear and finite for any $\phi \in \mathcal{D}$ since continuous functions on compact sets are bounded. In particular, we have:

$$|T(\phi)| = \left|\int_{-1}^{1} \phi(x)dx\right| \le \int_{-1}^{1} |\phi(x)|dx \le \sup_{x\in[-1,1]} |\phi(x)| \int_{-1}^{1} dx = 2 \sup_{x\in[-1,1]} |\phi(x)|,$$

which proves the continuity condition.

The definition in (2) is also linear in ϕ for the same reason as before. To show that the integral is finite for any $\phi \in \mathcal{D}$, we use the fact that ϕ is compactly supported and hence bounded. In particular, for any $\phi \in \mathcal{D}$, there exists a compact interval $[a, b] \subset \mathbb{R}$ such that $\text{supp}(\phi) \subset [a, b]$. Then we have:

$$|T(\phi)| = \left|\int_{-\infty}^{\infty} \phi(x)dx\right| \le \int_{a}^{b} |\phi(x)|dx \le \sup_{x\in[a,b]} |\phi(x)| \int_{a}^{b} dx = (b-a) \sup_{x\in[a,b]} |\phi(x)| < \infty,$$

which shows the continuity condition holds.

The definition in (3) is not a distribution since it is not linear in ϕ. There are many different ways to see this. For example, consider $\phi \in \mathcal{D}$ such that $\phi(x) \ge 0$ for all $x \in \mathbb{R}$. Set $\psi = -\phi \in \mathcal{D}$. Then we have:

$$S(\phi + \psi) = \int_{0}^{1} |\phi(x) + \psi(x)|dx = \int_{0}^{1} 0\,dx = 0,$$

but

$$S(\phi) + S(\psi) = \int_0^1 |\phi(x)|dx + \int_0^1 |\psi(x)|dx = 2\int_0^1 |\phi(x)|dx > 0.$$

Thus, $S(\phi+\psi) \neq S(\phi) + S(\psi)$, so S is nonlinear.

Exercise 3.
Let us observe that

$$\begin{aligned} 3 + \sin\left(\tfrac{3}{2}x\right)\sin\left(\tfrac{x}{2}\right) &= 3 + \frac{e^{i\frac{3}{2}x} - e^{-i\frac{3}{2}x}}{2i}\,\frac{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}}{2i} \\ &= 3 - \frac{1}{2}\,\frac{e^{i2x} - e^{ix} - e^{-ix} + e^{-i2x}}{2} \\ &= 3 + \frac{1}{2}\,\frac{e^{ix} - e^{-ix}}{2} - \frac{1}{2}\,\frac{e^{i2x} - e^{-i2x}}{2} \\ &= 3 + \frac{1}{2}\cos(x) - \frac{1}{2}\cos(2x), \end{aligned}$$

which is a (finite) Fourier series ($a_0 = 6,\ a_1 = \frac{1}{2},\ b_1 = 0,\ a_2 = -\frac{1}{2},\ b_2 = 0,\ \forall n \geq 3,\ a_n = b_n = 0$).

We look for u in the form of a Fourier series, that is, we set

$$u(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\Big(A_n\cos(nx) + B_n\sin(nx)\Big).$$

Then

$$\begin{aligned} u'(x) &= \sum_{n=1}^{\infty}\Big(nB_n\cos(nx) - nA_n\sin(nx)\Big), \\ u''(x) &= \sum_{n=1}^{\infty}\Big(-n^2A_n\cos(nx) - n^2B_n\sin(nx)\Big), \\ u(x+\pi) &= \frac{A_0}{2} + \sum_{n=1}^{\infty}\Big((-1)^nA_n\cos(nx) + (-1)^nB_n\sin(nx)\Big). \end{aligned}$$

Thus,

$$u''(x) - 2u(x+\pi) = -A_0 + \sum_{n=1}^{\infty}\Big(A_n(-n^2 - 2(-1)^n)\cos(nx) + B_n(-n^2 - 2(-1)^n)\sin(nx)\Big).$$

By matching term-by-term this Fourier series with the Fourier series of the right-hand side of the original equation, we obtain:

$$
\begin{aligned}
&n = 0 \quad && -A_0 = 3 \\
&n = 1 && \begin{cases} (-1+2)A_1 = \frac{1}{2}, \\ (-1+2)B_1 = 0, \end{cases} \\
&n = 2 && \begin{cases} (-4-2)A_2 = -\frac{1}{2}, \\ (-4-2)B_2 = 0, \end{cases} \\
&n \geq 3 && \begin{cases} (-n^2 - 2(-1)^n)A_n = 0, \\ (-n^2 - 2(-1)^n)B_n = 0. \end{cases}
\end{aligned}
$$

Hence we deduce

$$A_0 = -3, \qquad A_1 = \tfrac{1}{2}, \quad B_1 = 0, \qquad A_2 = \tfrac{1}{12}, \quad B_2 = 0, \qquad A_n = B_n = 0 \text{ for all } n \geq 3.$$

Thus the solution is

$$u(x) = -\frac{3}{2} + \frac{1}{2}\cos(x) + \frac{1}{12}\cos(2x).$$

Exercise 4.
For any test function $\phi \in \mathcal{D}$, we have by the definition of the distributional derivative:

$$
\begin{aligned}
\langle f', \phi \rangle = -\langle f, \phi' \rangle &= -\int_{-\infty}^{\infty} |x|\phi'(x)\,dx \\
&= -\int_{0}^{\infty} x\phi'(x)\,dx + \int_{-\infty}^{0} x\phi'(x)\,dx.
\end{aligned}
$$

We use integration by parts. Since ϕ is compactly supported, the boundary terms vanish ($[x\phi(x)]_0^\infty = 0$ and $[x\phi(x)]_{-\infty}^0 = 0$).

$$
\begin{aligned}
-\int_0^\infty x\phi'(x)\,dx &= -\left([x\phi(x)]_0^\infty - \int_0^\infty \phi(x)\,dx\right) = \int_0^\infty \phi(x)\,dx, \\
\int_{-\infty}^0 x\phi'(x)\,dx &= [x\phi(x)]_{-\infty}^0 - \int_{-\infty}^0 \phi(x)\,dx = -\int_{-\infty}^0 \phi(x)\,dx.
\end{aligned}
$$

Substituting these back, we find:

$$\langle f', \phi \rangle = \int_0^\infty \phi(x)\,dx - \int_{-\infty}^0 \phi(x)\,dx.$$

We define the function g by:

$$g(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Apparently, $\langle f', \phi \rangle = \langle g, \phi \rangle$.

Note that we can also write this in terms of the Heaviside step function $H(x)$, where

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

One easily sees that $g(x) = 2H(x) - 1$ is an alternative way of writing the distributional derivative.

Exercise 5.
Step 1 (Separation of variables). This exercise corresponds to Example 18.1, with $L = \pi$, $a = 1$, and $f(x) = \cos x - \cos(3x)$. We have seen that the general solution of the equation is

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin(nx)\, e^{-n^2 t}.$$

Step 2 (Initial condition). We still need to choose the coefficients α_n so that

$$u(x,0) = f(x) = \cos x - \cos(3x) = \sum_{n=1}^{\infty} \alpha_n \sin(nx).$$

We must therefore have

$$\begin{aligned} \alpha_n &= \frac{2}{\pi} \int_0^{\pi} [\cos x \,\sin(nx) - \cos(3x)\, \sin(nx)]\, dx \\ &= \frac{1}{\pi} \int_0^{\pi} \big[\sin((n+1)x) + \sin((n-1)x) - \sin((n+3)x) - \sin((n-3)x)\big]\, dx. \end{aligned}$$

If $n \neq 1$ and $n \neq 3$, then

$$\begin{aligned} \alpha_n &= \frac{1}{\pi} \left[-\frac{\cos((n+1)x)}{n+1} - \frac{\cos((n-1)x)}{n-1} + \frac{\cos((n+3)x)}{n+3} + \frac{\cos((n-3)x)}{n-3} \right]_0^{\pi} \\ &= \frac{1}{\pi} \left[\frac{1-(-1)^{n+1}}{n+1} + \frac{1-(-1)^{n-1}}{n-1} - \frac{1-(-1)^{n+3}}{n+3} - \frac{1-(-1)^{n-3}}{n-3} \right]. \end{aligned}$$

Thus $\alpha_n = 0$ if n is odd ($n \neq 1$ and $n \neq 3$). If n is even, then

$$\begin{aligned}\alpha_n &= \frac{2}{\pi}\left[\frac{1}{n+1} + \frac{1}{n-1} - \frac{1}{n+3} - \frac{1}{n-3}\right] \\ &= \frac{4n}{\pi}\left[\frac{1}{n^2-1} - \frac{1}{n^2-9}\right] = \frac{-32n}{\pi(n^2-1)(n^2-9)}.\end{aligned}$$

Moreover,

$$\begin{aligned}\alpha_1 &= \frac{1}{\pi}\int_0^\pi [\sin(2x) - \sin(4x) + \sin(2x)]\,dx = 0, \\ \alpha_3 &= \frac{1}{\pi}\int_0^\pi [\sin(4x) + \sin(2x) - \sin(6x)]\,dx = 0.\end{aligned}$$

We thus obtain the solution

$$u(x,t) = \frac{-64}{\pi}\sum_{k=1}^{\infty} \frac{k}{(4k^2-1)(4k^2-9)}\,\sin(2kx)\,e^{-4k^2 t}.$$

Exercise 6.
We begin by finding solutions of

$$\begin{cases} \Delta u = 0 & x, y \in (0,\pi) \\ \dfrac{\partial u}{\partial y}(x,0) = \dfrac{\partial u}{\partial y}(x,\pi) = 0 & x \in (0,\pi) \end{cases}$$

of the form $u(x,y) = v(x)w(y)$. We have

$$\begin{cases} v''(x)w(y) + v(x)w''(y) = 0 \quad \Leftrightarrow \quad -\dfrac{v''(x)}{v(x)} = -\lambda = \dfrac{w''(y)}{w(y)}, \\ \dfrac{\partial}{\partial y}u(x,0) = v(x)w'(0) = \dfrac{\partial}{\partial y}u(x,\pi) = v(x)w'(\pi) = 0, \end{cases}$$

that is,

$$\begin{cases} w''(y) + \lambda w(y) = 0 \\ w'(0) = w'(\pi) = 0 \end{cases} \tag{18.28}$$

and

$$v''(x) - \lambda v(x) = 0. \tag{18.29}$$

We have seen (cf. example 17.4) that the nontrivial solutions of (18.28) are given by $\lambda = n^2$ with $n = 0, 1, 2, \ldots$ and $w_n(y) = a_n\cos(ny)$ (for $n = 0$, $w_0(y) = a_0/2$). Equation (18.29) then becomes

$$v_n''(x) - n^2 v_n(x) = 0 \quad \Rightarrow \quad v_n(x) = \begin{cases} b_1 x + b_0, & \text{if } n = 0, \\ b_n\cosh(nx) + c_n\sinh(nx), & \text{if } n \neq 0. \end{cases}$$

Therefore the general solution of the given equation is

$$u(x,y) = \frac{a_0}{2}\big(b_1 x + b_0\big) + \sum_{n=1}^{\infty} a_n \big(b_n \cosh(nx) + c_n \sinh(nx)\big)\cos(ny),$$

or, writing $\frac{a_0}{2}b_1 = \alpha,\ \frac{a_0}{2}b_0 = \frac{\beta}{2},\ a_n b_n = A_n,\ a_n c_n = B_n,$

$$u(x,y) = \alpha x + \frac{\beta}{2} + \sum_{n=1}^{\infty}\big(A_n \cosh(nx) + B_n \sinh(nx)\big)\cos(ny).$$

Step 2 (Boundary conditions). We further require

$$u(0,y) = \cos(2y) \quad \text{and} \quad u(\pi,y) = 0,$$

that is,

$$\frac{\beta}{2} + \sum_{n=1}^{\infty} A_n \cos(ny) = \cos(2y),$$

$$\alpha\pi + \frac{\beta}{2} + \sum_{n=1}^{\infty}\big(A_n \cosh(n\pi) + B_n \sinh(n\pi)\big)\cos(ny) = 0.$$

The first equation immediately gives $\beta = 0$ and $A_n = 0$ for $n \neq 2$ and $A_2 = 1$. Substituting this into the second equation yields $\alpha = 0$ and

$$A_n \cosh(n\pi) + B_n \sinh(n\pi) = 0.$$

Hence if $n \neq 2$ then $A_n = B_n = 0$, whereas for $n = 2$ we have

$$B_2 = -\frac{\cosh(2\pi)}{\sinh(2\pi)}.$$

Therefore the solution of the problem is

$$u(x,y) = \left[\cosh(2x) - \frac{\cosh(2\pi)}{\sinh(2\pi)}\sinh(2x)\right]\cos(2y).$$