

Exercise 1.

The condition $y(x + 2\pi) = y(x)$ tells us that we are looking for a 2π -periodic solution, so we seek it in the form of a Fourier series:

$$\begin{aligned}
 y(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \\
 y'(x) &= \sum_{n=1}^{\infty} (nb_n \cos(nx) - na_n \sin(nx)) \\
 y''(x) &= \sum_{n=1}^{\infty} (-n^2 a_n \cos(nx) - n^2 b_n \sin(nx)) \\
 y'(x - \pi) &= \sum_{n=1}^{\infty} (nb_n \cos(n(x - \pi)) - na_n \sin(n(x - \pi))) \\
 &= \sum_{n=1}^{\infty} ((-1)^n nb_n \cos(nx) - (-1)^n na_n \sin(nx))
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 &y''(x) + 5y'(x - \pi) - y(x) \\
 &= -\frac{a_0}{2} + \sum_{n=1}^{\infty} [(-a_n + 5(-1)^n nb_n - n^2 a_n) \cos(nx) + (-b_n - 5(-1)^n na_n - n^2 b_n) \sin(nx)].
 \end{aligned}$$

Now, the right-hand side of the equation is given as a Fourier series (in fact a finite sum), and by equating coefficients term by term, we obtain the systems:

$$\begin{aligned}
 \boxed{n = 0} & \quad -\frac{a_0}{2} = 2 \\
 \boxed{n = 1} & \quad \begin{cases} -a_1 - 5b_1 - a_1 = 1 \\ -b_1 + 5a_1 - b_1 = 0 \end{cases} \\
 \boxed{n = 2} & \quad \begin{cases} -a_2 + 10b_2 - 4a_2 = 0 \\ -b_2 - 10a_2 - 4b_2 = -3 \end{cases} \\
 \boxed{n \geq 3} & \quad \begin{cases} -a_n + 5(-1)^n nb_n - n^2 a_n = 0 \\ -b_n - 5(-1)^n na_n - n^2 b_n = 0 \end{cases}
 \end{aligned}$$

whose solutions are

$$\begin{aligned}
 a_0 &= -4 \\
 a_1 &= -\frac{2}{29} \\
 b_1 &= -\frac{5}{29} \\
 a_2 &= \frac{6}{25} \\
 b_2 &= \frac{3}{25} \\
 a_n &= 0 \quad \forall n \geq 3 \\
 b_n &= 0 \quad \forall n \geq 3.
 \end{aligned}$$

Thus,

$$y(x) = -2 - \frac{2}{29} \cos(x) - \frac{5}{29} \sin(x) + \frac{6}{25} \cos(2x) + \frac{3}{25} \sin(2x)$$

Exercise 2.

Applying the Fourier transform inversion theorem for the function f defined by $f(x) = xe^{-\omega|x|}$ (i.e., it is continuous) we obtain:

$$\begin{aligned}
 xe^{-\omega|x|} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathfrak{F}(f)(\alpha) e^{i\alpha x} d\alpha = \frac{4\omega}{2\pi i} \int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2 + \omega^2)^2} e^{i\alpha x} d\alpha \\
 &= \frac{2\omega}{\pi i} \left[\int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2 + \omega^2)^2} \cos(\alpha x) d\alpha + i \int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2 + \omega^2)^2} \sin(\alpha x) d\alpha \right].
 \end{aligned}$$

As $f(x) = xe^{-\omega|x|} \in \mathbb{R}$, we get:

$$\begin{aligned}
 \int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2 + \omega^2)^2} \cos(\alpha x) d\alpha &= 0, \\
 \int_{-\infty}^{+\infty} \frac{\alpha}{(\alpha^2 + \omega^2)^2} \sin(\alpha x) d\alpha &= \frac{\pi}{2\omega} xe^{-\omega|x|}.
 \end{aligned}$$

The function g defined by $g(\alpha) = \frac{\alpha}{(\alpha^2 + \omega^2)^2} \sin(\alpha x)$ is even, then we obtain:

$$\int_0^{+\infty} \frac{\alpha}{(\alpha^2 + \omega^2)^2} \sin(\alpha x) d\alpha = \frac{\pi}{4\omega} xe^{-\omega|x|}.$$

Evaluating it at $\alpha = t$ and choosing $\omega = 2$ and $x = \frac{1}{2}$ we obtain:

$$\int_0^{+\infty} \frac{t}{(t^2 + 4)^2} \sin\left(\frac{t}{2}\right) dt = \frac{\pi}{16e}.$$

Exercise 3.

Using the convolution product notation, our equation is written as

$$y + 3y * f = f$$

Taking the Fourier transform:

$$\begin{aligned}\hat{y} + 3\widehat{y * f} &= \hat{f} \\ \hat{y} + 3\sqrt{2\pi}\hat{y}\hat{f} &= \hat{f} \\ \hat{y}(1 + 3\sqrt{2\pi}\hat{f}) &= \hat{f}\end{aligned}$$

Using the hint with $\omega = 1$, we find the Fourier transform of f :

$$\begin{aligned}\hat{y} &= \frac{\hat{f}}{1 + 3\sqrt{2\pi}\hat{f}} \\ &= \frac{\sqrt{\frac{2}{\pi}}\frac{1}{\alpha^2+1}}{1 + 3\sqrt{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{\alpha^2+1}} \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{1 + \alpha^2 + 6} \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{\alpha^2 + 7}\end{aligned}$$

Thus, using the hint with $\omega = \sqrt{7}$, we obtain $y(x) = \frac{1}{\sqrt{7}}e^{-\sqrt{7}|x|}$.

Exercise 4.

Writing the equation using the convolution product, we get

$$3y + (y'' - y) * f = g$$

Taking the Fourier transform, we arrive at

$$\begin{aligned}3\hat{y} + \widehat{(y'' - y) * f} &= \hat{g} \\ 3\hat{y} + \sqrt{2\pi}(\widehat{y'' - y})\hat{f} &= \hat{g} \\ 3\hat{y} + \sqrt{2\pi}((i\alpha)^2\hat{y} - \hat{y})\sqrt{\frac{2}{\pi}}\frac{1}{\alpha^2 + 1} &= \hat{g} \\ 3\hat{y} - 2(\alpha^2 + 1)\hat{y}\frac{1}{\alpha^2 + 1} &= \hat{g} \\ 3\hat{y} - 2\hat{y} &= \hat{g} \\ \hat{y} &= \hat{g}\end{aligned}$$

and therefore $y(x) = g(x) = xe^{-x^2}$