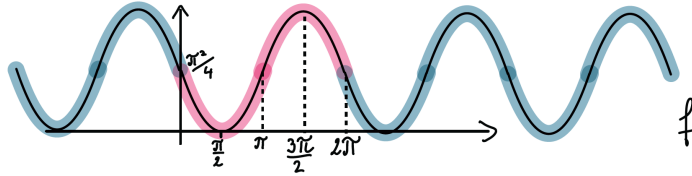


**Exercise 1.**

We begin by computing the Fourier series of  $f$ .



$$\begin{aligned}
 a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) dx \\
 &= \frac{1}{\pi} \int_0^{\pi} \left(x - \frac{\pi}{2}\right)^2 dx + \frac{1}{\pi} \int_{\pi}^{2\pi} \left(\frac{\pi^2}{2} - \left(x - \frac{3\pi}{2}\right)^2\right) dx \\
 &= \frac{1}{\pi} \left[ \frac{1}{3} \left(x - \frac{\pi}{2}\right)^3 \right]_{x=0}^{x=\pi} + \frac{\pi^2}{2} - \frac{1}{\pi} \left[ \frac{1}{3} \left(x - \frac{3\pi}{2}\right)^3 \right]_{x=\pi}^{x=2\pi} \\
 &= \frac{1}{3\pi} \left( \frac{\pi^3}{8} - \frac{-\pi^3}{8} \right) + \frac{\pi^2}{2} - \frac{1}{3\pi} \left( \frac{\pi^3}{8} - \frac{-\pi^3}{8} \right) \\
 &= \frac{\pi^2}{2}.
 \end{aligned}$$

and for  $n \geq 1$ ,

*Variant 1:* with a small clever change of variable.

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_0^{\pi} \left(x - \frac{\pi}{2}\right)^2 \cos(nx) dx + \frac{1}{\pi} \int_{\pi}^{2\pi} \left(\frac{\pi^2}{2} - \left(x - \frac{3\pi}{2}\right)^2\right) \cos(nx) dx \\
 &\stackrel{y=x-\pi}{=} \frac{1}{\pi} \int_0^{\pi} \left(x - \frac{\pi}{2}\right)^2 \cos(nx) dx + \frac{\pi}{2} \int_{\pi}^{2\pi} \cos(nx) dx - \frac{1}{\pi} \int_0^{\pi} \left(y - \frac{\pi}{2}\right)^2 \underbrace{\cos(ny - n\pi)}_{=(-1)^n \cos(ny)} dy \\
 &= \frac{1 - (-1)^n}{\pi} \underbrace{\int_0^{\pi} \left(x - \frac{\pi}{2}\right)^2 \cos(nx) dx}_{:=I_n} + \frac{\pi}{2} \underbrace{\left[ \frac{\sin(nx)}{n} \right]_{x=\pi}^{x=2\pi}}_{=0}
 \end{aligned}$$

Compute  $I_n$  using IBP (IPP)

$$\begin{aligned}
I_n &\stackrel{\text{IPP}}{=} \left[ \left(x - \frac{\pi}{2}\right)^2 \frac{1}{n} \sin(nx) \right]_{x=0}^{x=\pi} - \frac{2}{n} \int_0^\pi \left(x - \frac{\pi}{2}\right) \sin(nx) dx \quad \left| \begin{array}{l} u = \left(x - \frac{\pi}{2}\right)^2 \quad v = \frac{1}{n} \sin(nx) \\ u' = 2\left(x - \frac{\pi}{2}\right) \quad v' = \cos(nx) \end{array} \right. \\
&= -\frac{2}{n} \int_0^\pi \left(x - \frac{\pi}{2}\right) \sin(nx) dx \\
&\stackrel{\text{IPP}}{=} -\frac{2}{n} \left[ \left(x - \frac{\pi}{2}\right) \frac{-1}{n} \cos(nx) \right]_{x=0}^{x=\pi} - \frac{2}{n^2} \int_0^\pi \cos(nx) dx \quad \left| \begin{array}{l} u = \left(x - \frac{\pi}{2}\right) \quad v = \frac{-1}{n} \cos(nx) \\ u' = 1 \quad v' = \sin(nx) \end{array} \right. \\
&= \frac{\pi}{n^2} \underbrace{\cos(n\pi)}_{(-1)^n} - \frac{-\pi}{n^2} - \frac{2}{n^2} \underbrace{\left[ \frac{1}{n} \sin(nx) \right]_{x=0}^{x=\pi}}_{=0} \\
&= \frac{\pi}{n^2} (1 + (-1)^n)
\end{aligned}$$

To finish,

$$a_n = \frac{(1 - (-1)^n)(1 + (-1)^n)}{n^2} = \frac{1 - (-1)^{2n}}{n^2} = 0$$

Same trick for  $b_n$ :

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_0^\pi \left(x - \frac{\pi}{2}\right)^2 \sin(nx) dx + \frac{1}{\pi} \int_\pi^{2\pi} \left(\frac{\pi^2}{2} - \left(x - \frac{3\pi}{2}\right)^2\right) \sin(nx) dx \\
&\stackrel{y=x-\pi}{=} \frac{1}{\pi} \int_0^\pi \left(x - \frac{\pi}{2}\right)^2 \sin(nx) dx + \frac{\pi}{2} \int_\pi^{2\pi} \sin(nx) dx - \frac{1}{\pi} \int_0^\pi \left(y - \frac{\pi}{2}\right)^2 \underbrace{\sin(ny - n\pi)}_{=(-1)^n \sin(ny)} dy \\
&= \frac{1 - (-1)^n}{\pi} \underbrace{\int_0^\pi \left(x - \frac{\pi}{2}\right)^2 \sin(nx) dx}_{:=J_n} + \frac{\pi}{2} \left[ \frac{-\cos(nx)}{n} \right]_{x=\pi}^{x=2\pi} \\
&= \frac{1 - (-1)^n}{\pi} J_n + \frac{\pi((-1)^n - 1)}{2n} = (1 - (-1)^n) \left( \frac{1}{\pi} J_n - \frac{\pi}{2n} \right)
\end{aligned}$$

Finally compute  $J_n$  with IBP (IPP)

$$\begin{aligned}
J_n &= \int_0^\pi \left(x - \frac{\pi}{2}\right)^2 \sin(nx) dx \\
&\stackrel{\text{IPP}}{=} \left[ \left(x - \frac{\pi}{2}\right)^2 \frac{-1}{n} \cos(nx) \right]_{x=0}^{x=\pi} + \frac{2}{n} \int_0^\pi \left(x - \frac{\pi}{2}\right) \cos(nx) dx \quad \left| \begin{array}{l} u = \left(x - \frac{\pi}{2}\right)^2 \quad v = \frac{-1}{n} \cos(nx) \\ u' = 2 \left(x - \frac{\pi}{2}\right) \quad v' = \sin(nx) \end{array} \right. \\
&= \frac{\pi^2}{4n} (1 - (-1)^n) + \frac{2}{n} \int_0^\pi \left(x - \frac{\pi}{2}\right) \cos(nx) dx \\
&\stackrel{\text{IPP}}{=} \frac{\pi^2}{4n} (1 - (-1)^n) \\
&\quad + \frac{2}{n} \underbrace{\left[ \left(x - \frac{\pi}{2}\right) \frac{1}{n} \sin(nx) \right]_{x=0}^{x=\pi}}_{=0} - \frac{2}{n^2} \int_0^\pi \sin(nx) dx \quad \left| \begin{array}{l} u = \left(x - \frac{\pi}{2}\right) \quad v = \frac{1}{n} \sin(nx) \\ u' = 1 \quad v' = \cos(nx) \end{array} \right. \\
&= \frac{\pi^2}{4n} (1 - (-1)^n) + \frac{2}{n^3} [\cos(nx)]_{x=0}^{x=\pi} \\
&= \frac{\pi^2}{4n} (1 - (-1)^n) + \frac{2((-1)^n - 1)}{n^3} \\
&= (1 - (-1)^n) \left( \frac{\pi^2}{4n} - \frac{2}{n^3} \right)
\end{aligned}$$

To conclude,

$$\begin{aligned}
b_n &= (1 - (-1)^n) \left( (1 - (-1)^n) \left( \frac{\pi}{4n} - \frac{2}{\pi n^3} \right) - \frac{\pi}{2n} \right) \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 \left( 2 \left( \frac{\pi}{4(2k-1)} - \frac{2}{\pi(2k-1)^3} \right) - \frac{\pi}{2(2k-1)} \right) & \text{if } n = 2k - 1 \text{ with } k \in \mathbb{N}^* \end{cases} \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-8}{\pi(2k-1)^3} & \text{if } n = 2k - 1 \text{ with } k \in \mathbb{N}^* \end{cases}
\end{aligned}$$

*Variant 2:* Brute force.

We start by computing some antiderivatives. For  $\alpha \neq 0$ ,

$$\begin{aligned} \int^x t \cos(\alpha t) dt &\stackrel{\text{IPP}}{=} \frac{1}{\alpha} x \sin(\alpha x) - \frac{1}{\alpha} \int^x \sin(\alpha t) dt && \left| \begin{array}{l} u = t \quad v = \frac{1}{\alpha} \sin \alpha t \\ u' = 1 \quad v' = \cos \alpha t \end{array} \right. \\ &= \frac{1}{\alpha} x \sin(\alpha x) + \frac{1}{\alpha^2} \cos(\alpha x) \\ \int^x t \sin(\alpha t) dt &\stackrel{\text{IPP}}{=} -\frac{1}{\alpha} x \cos(\alpha x) + \frac{1}{\alpha} \int^x \cos(\alpha t) dt && \left| \begin{array}{l} u = t \quad v = -\frac{1}{\alpha} \cos(\alpha t) \\ u' = 1 \quad v' = \sin(\alpha t) \end{array} \right. \\ &= -\frac{1}{\alpha} x \cos(\alpha x) + \frac{1}{\alpha^2} \sin(\alpha x) \\ \int^x t^2 \cos(\alpha t) dt &\stackrel{\text{IPP}}{=} \frac{1}{\alpha} x^2 \sin(\alpha x) - \frac{2}{\alpha} \int^x t \sin(\alpha t) dt && \left| \begin{array}{l} u = t^2 \quad v = \frac{1}{\alpha} \sin(\alpha t) \\ u' = 2t \quad v' = \cos(\alpha t) \end{array} \right. \\ &= \frac{x^2}{\alpha} \sin(\alpha x) + \frac{2x}{\alpha^2} \cos(\alpha x) - \frac{2}{\alpha^3} \sin(\alpha x) \\ \int^x t^2 \sin(\alpha t) dt &\stackrel{\text{IPP}}{=} -\frac{1}{\alpha} x^2 \cos(\alpha x) + \frac{2}{\alpha} \int^x t \cos(\alpha t) dt && \left| \begin{array}{l} u = t^2 \quad v = -\frac{1}{\alpha} \cos(\alpha t) \\ u' = 2t \quad v' = \sin(\alpha t) \end{array} \right. \\ &= -\frac{x^2}{\alpha} \cos(\alpha x) + \frac{2x}{\alpha^2} \sin(\alpha x) + \frac{2}{\alpha^3} \cos(\alpha x) \end{aligned}$$

Thus

$$\begin{aligned} \int_0^\pi \left(x - \frac{\pi}{2}\right)^2 \cos(nx) dx &= \int_0^\pi x^2 \cos(nx) - \pi \int_0^\pi x \cos(nx) dx + \frac{\pi^2}{4} \int_0^\pi \cos(nx) dx \\ &= \left[ \frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) - \frac{2}{n^3} \sin(nx) \right]_{x=0}^{x=\pi} \\ &\quad - \pi \left[ \frac{1}{n} x \sin(nx) + \frac{1}{n^2} \cos(nx) \right]_{x=0}^{x=\pi} \\ &\quad + \frac{\pi^2}{4} \left[ \frac{1}{n^2} \sin(nx) \right]_{x=0}^{x=\pi} \\ &= \frac{2\pi}{n^2} \cos(n\pi) - \pi \frac{1}{n^2} (\cos(n\pi) - 1) \\ &= \frac{2\pi}{n^2} (-1)^n - \frac{\pi}{n^2} (-1)^n + \frac{\pi}{n^2} \\ &= \frac{\pi}{n^2} ((-1)^n + 1) \end{aligned}$$

and

$$\begin{aligned}
\int_{\pi}^{2\pi} \left( \frac{\pi^2}{2} - \left( x - \frac{3\pi}{2} \right)^2 \right) \cos(nx) dx &= - \int_{\pi}^{2\pi} x^2 \cos(nx) dx + 3\pi \int_{\pi}^{2\pi} x \cos(nx) dx - \frac{7\pi^2}{4} \int_{\pi}^{2\pi} \cos(nx) dx \\
&= - \left[ \frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) - \frac{2}{n^3} \sin(nx) \right]_{x=\pi}^{x=2\pi} \\
&\quad + 3\pi \left[ \frac{1}{n} x \sin(nx) + \frac{1}{n^2} \cos(nx) \right]_{x=\pi}^{x=2\pi} \\
&\quad - \frac{7\pi^2}{4} \left[ \frac{1}{n^2} \sin(nx) \right]_{x=\pi}^{2\pi} \\
&= - \frac{2\pi}{n^2} (2 - \cos(n\pi)) + \frac{3\pi}{n^2} (1 - \cos(n\pi)) \\
&= - \frac{\pi}{n^2} - \frac{\pi}{n^2} (-1)^n \\
&= - \frac{\pi}{n^2} (1 + (-1)^n)
\end{aligned}$$

And so,

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_0^{\pi} \left( x - \frac{\pi}{2} \right)^2 \cos(nx) dx + \frac{1}{\pi} \int_{\pi}^{2\pi} \left( \frac{\pi^2}{2} - \left( x - \frac{3\pi}{2} \right)^2 \right) \cos(nx) dx \\
&= \frac{1}{n^2} (1 + (-1)^n) - \frac{1}{n^2} (1 + (-1)^n) = 0
\end{aligned}$$

Next,

$$\begin{aligned}
\int_0^{\pi} \left( x - \frac{\pi}{2} \right)^2 \sin(nx) dx &= \int_0^{\pi} x^2 \sin(nx) dx - \pi \int_0^{\pi} x \sin(nx) dx + \frac{\pi^2}{4} \int_0^{\pi} \sin(nx) dx \\
&= \left[ -\frac{x^2}{n} \cos(nx) + \frac{2x}{n^2} \sin(nx) + \frac{2}{n^3} \cos(nx) \right]_{x=0}^{x=\pi} \\
&\quad - \pi \left[ -\frac{1}{n} x \cos(nx) + \frac{1}{n^2} \sin(nx) \right]_{x=0}^{x=\pi} \\
&\quad + \frac{\pi^2}{4} \left[ -\frac{1}{n} \cos(nx) \right]_{x=0}^{x=\pi} \\
&= - \frac{\pi^2}{n} \cos(n\pi) + \frac{2}{n^3} (\cos(n\pi) - 1) + \frac{\pi^2}{n} \cos(n\pi) - \frac{\pi^2}{4n} (\cos(n\pi) - 1) \\
&= \frac{2}{n^3} ((-1)^n - 1) + \frac{\pi^2}{4n} (1 - (-1)^n)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\pi}^{2\pi} \left( \frac{\pi^2}{2} - \left( x - \frac{3\pi}{2} \right)^2 \right) \sin(nx) dx &= - \int_{\pi}^{2\pi} x^2 \sin(nx) + 3\pi \int_{\pi}^{2\pi} x \sin(nx) dx - \frac{7\pi^2}{4} \int_{\pi}^{2\pi} \sin(nx) dx \\
&= - \left[ -\frac{x^2}{n} \cos(nx) + \frac{2x}{n^2} \sin(nx) + \frac{2}{n^3} \cos(nx) \right]_{x=\pi}^{x=2\pi} \\
&\quad + 3\pi \left[ -\frac{1}{n} x \cos(nx) + \frac{1}{n^2} \sin(nx) \right]_{x=\pi}^{x=2\pi} \\
&\quad - \frac{7\pi^2}{4} \left[ -\frac{1}{n} \cos(nx) \right]_{x=\pi}^{x=2\pi} \\
&= \frac{\pi^2}{n} (4 - \cos(n\pi)) - \frac{2}{n^3} (1 - \cos(n\pi)) - \frac{3\pi^2}{n} (2 - \cos(n\pi)) \\
&\quad + \frac{7\pi^2}{4n} (1 - \cos(n\pi)) \\
&= \frac{\pi^2}{n} \left( \frac{7}{4} + 4 - 6 \right) + \frac{\pi^2}{n} (-1)^n \left( -1 + 3 - \frac{7}{4} \right) + \frac{2}{n^3} ((-1)^n - 1) \\
&= \frac{\pi^2}{4n} ((-1)^n - 1) + \frac{2}{n^3} ((-1)^n - 1)
\end{aligned}$$

And therefore,

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_0^{\pi} \left( x - \frac{\pi}{2} \right)^2 \sin(nx) dx + \frac{1}{\pi} \int_{\pi}^{2\pi} \left( \frac{\pi^2}{2} - \left( x - \frac{3\pi}{2} \right)^2 \right) \sin(nx) dx \\
&= \frac{4}{\pi n^3} ((-1)^n - 1) \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-8}{\pi(2k-1)^3} & \text{if } n = 2k - 1 \text{ with } k \in \mathbb{N}^* \end{cases}
\end{aligned}$$

Now that we know the Fourier coefficients of  $f$ , we look for a solution of the differential equation in the form of a Fourier series:

$$\begin{aligned}
y(x) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos(nx) + B_n \sin(nx)) \\
y'(x) &= \sum_{n=1}^{\infty} (nB_n \cos(nx) - nA_n \sin(nx)) \\
y'(x) + y(x) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} \{ (A_n + nB_n) \cos(nx) + (B_n - nA_n) \sin(nx) \} \\
&= f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))
\end{aligned}$$

We thus obtain the family of systems

$$\boxed{n = 0} \quad A_0 = a_0 = \frac{\pi^2}{2}$$

$$\boxed{n \geq 1} \quad \begin{cases} A_n + nB_n = a_n = 0 \\ B_n - nA_n = b_n \end{cases} \Rightarrow \begin{cases} A_n = \frac{-n}{n^2 + 1} b_n \\ B_n = \frac{1}{n^2 + 1} b_n \end{cases}$$

and therefore

$$A_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{\pi(2k-1)^2((2k-1)^2+1)} & \text{if } n = 2k - 1 \text{ with } k \in \mathbb{N}^* \end{cases}$$

$$B_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-8}{\pi(2k-1)^3((2k-1)^2+1)} & \text{if } n = 2k - 1 \text{ with } k \in \mathbb{N}^* \end{cases}$$

To finish,

$$y(x) = \frac{\pi^2}{4} + \sum_{k=1}^{\infty} \left( \frac{8 \cos((2k-1)x)}{\pi(2k-1)^2((2k-1)^2+1)} - \frac{8 \sin((2k-1)x)}{\pi(2k-1)^3((2k-1)^2+1)} \right)$$

Remark: The given equation is a first-order linear differential equation with constant coefficients. In Analysis II we saw that, for example by the integrating factor method, one can find the general solution as follows:

$$\begin{aligned} y'(x) + y(x) &= f(x) \\ y'(x)e^x + y(x)e^x &= f(x)e^x \\ \frac{d}{dx} [y(x)e^x] &= f(x)e^x \\ y(x)e^x &= \int_0^x f(t)e^t dt + C \\ y(x) &= e^{-x} \int_0^x f(t)e^t dt + Ce^{-x} \end{aligned}$$

To answer the exercise question, one must determine whether there exists a constant  $C$  such that the  $y$  given above is  $2\pi$ -periodic, knowing that  $f$  is.

It turns out this is possible!

We have

$$\begin{aligned}
y(x+2\pi) &= e^{-x-2\pi} \int_0^{x+2\pi} f(t)e^t dt + Ce^{-x-2\pi} \\
&= e^{-x-2\pi} \left( \int_0^{2\pi} f(t)e^t dt + \int_{2\pi}^{x+2\pi} f(t)e^t dt \right) + Ce^{-x-2\pi} \\
&\stackrel{t=s+2\pi}{=} e^{-x-2\pi} \left( \int_0^{2\pi} f(t)e^t dt + \int_0^x f(s+2\pi)e^{s+2\pi} ds \right) + Ce^{-x-2\pi} \\
&= e^{-x-2\pi} \left( \int_0^{2\pi} f(t)e^t dt + e^{2\pi} \int_0^x f(s)e^s ds \right) + Ce^{-x-2\pi} \\
&= e^{-x} \int_0^x f(s)e^s ds + Ce^{-x} \\
&\quad - Ce^{-x} + e^{-x-2\pi} \int_0^{2\pi} f(t)e^t dt + Ce^{-x-2\pi} \\
&= y(x) + e^{-x} \left( C(e^{-2\pi} - 1) + e^{-2\pi} \int_0^{2\pi} f(t)e^t dt \right)
\end{aligned}$$

Thus choosing

$$C = \frac{-e^{-2\pi}}{e^{-2\pi} - 1} \int_0^{2\pi} f(t)e^t dt = \frac{1}{e^{2\pi} - 1} \int_0^{2\pi} f(t)e^t dt$$

which is independent of  $x$ , we obtain that  $y$  is periodic.

Note further that the work done with Fourier series is the analogue of the method of undetermined coefficients<sup>1</sup> with a parameter  $n$  running around.

### Exercise 2.

1. Since  $f$  is even, we have  $b_n = 0$  for all  $n$ . Moreover,

$$a_0 = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \left[ -\frac{1}{3} \cos(3x) \right]_{x=0}^{x=\pi} = \frac{4}{3\pi}.$$

For  $n \geq 1$ ,

$$a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} \sin(3x) \cos(nx) dx.$$

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<sup>1</sup>Sometimes called the "I guess the shape of the solution" method. See [https://sma.epfl.ch/~struett/analyse2/main\\_proofs\\_at\\_the\\_end.pdf#theorem.1.26](https://sma.epfl.ch/~struett/analyse2/main_proofs_at_the_end.pdf#theorem.1.26)

Using Euler's formulas, we obtain

$$\sin(3x) \cos(nx) = \frac{e^{i3x} - e^{-i3x}}{2i} \frac{e^{inx} + e^{-inx}}{2} = \frac{1}{2} (\sin((n+3)x) - \sin((n-3)x)).$$

Thus, for  $n = 3$ ,

$$a_3 = \frac{1}{\pi} \int_0^\pi \sin(6x) dx = \frac{1}{\pi} \left[ -\frac{1}{6} \cos(6x) \right]_0^\pi = \frac{1}{\pi} \left( -\frac{1}{6} + \frac{1}{6} \right) = 0.$$

For  $n \neq 3$ ,

$$\begin{aligned} a_n &= \frac{1}{\pi} \left[ -\frac{1}{n+3} \cos((n+3)x) + \frac{1}{n-3} \cos((n-3)x) \right]_0^\pi \\ &= \frac{1}{\pi} \left( -\frac{1}{n+3} (-1)^{n+3} + \frac{1}{n-3} (-1)^{n-3} + \frac{1}{n+3} - \frac{1}{n-3} \right) \\ &= \frac{1}{\pi} \left( \frac{1 + (-1)^n}{n+3} - \frac{1 + (-1)^n}{n-3} \right) \\ &= \frac{1 + (-1)^n}{\pi} \frac{n-3 - n-3}{n^2 - 9} = -\frac{6}{\pi} \frac{1 + (-1)^n}{n^2 - 9}. \end{aligned}$$

Hence,

$$a_n = \begin{cases} -\frac{12}{\pi(4k^2 - 9)} & \text{if } n = 2k, k \geq 1, \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

The Fourier series of  $f$  is therefore

$$Ff(x) = \frac{2}{3\pi} - \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2 - 9}.$$

2. Since we are looking for an even solution, we seek it in the form

$$y(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx).$$

We have

$$y(x - \pi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx - n\pi) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (-1)^n A_n \cos(nx),$$

and thus,

$$y(x) - 2y(x - \pi) = -\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n (1 - 2(-1)^n) \cos(nx).$$

The coefficients  $A_n$  must therefore satisfy:

$$\boxed{n = 0} \quad -\frac{A_0}{2} = -\frac{2}{3\pi},$$

$$\boxed{n = 2k} \quad -A_n = -\frac{12}{\pi(4k^2 - 9)},$$

$$\boxed{n = 2k + 1} \quad 3A_n = 0.$$

Finally,

$$y(x) = \frac{-2}{3\pi} + \frac{12}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2 - 9}.$$

### Exercise 3.

We look for a solution of the form

$$y(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).$$

We then compute

$$\begin{aligned} y(x - \pi) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx - n\pi) + b_n \sin(nx - n\pi)) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n (-1)^n \cos(nx) + b_n (-1)^n \sin(nx)), \end{aligned}$$

and therefore

$$\begin{aligned} y(x) + 2y(x - \pi) &= \frac{3}{2}a_0 + \sum_{n=1}^{\infty} \{(1 + 2(-1)^n) a_n \cos(nx) + (1 + 2(-1)^n) b_n \sin(nx)\} \\ &= \cos(x) + 3 \sin(2x) + 4 \cos(5x). \end{aligned}$$

Moreover, if  $A_n$  and  $B_n$  denote the Fourier coefficients of  $\cos(x) + 3 \sin(2x) + 4 \cos(5x)$ , then  $A_1 = 1$ ,  $A_5 = 4$ ,  $B_2 = 3$ , and all other coefficients are zero.

Thus, we obtain the systems

$$\boxed{n = 0} \quad \frac{3}{2}a_0 = 0,$$

$$\boxed{n = 1} \quad \begin{cases} -1 a_1 = 1, \\ -1 b_1 = 0, \end{cases}$$

$$\boxed{n = 2} \quad \begin{cases} 3a_2 = 0, \\ 3b_2 = 3, \end{cases}$$

$$\boxed{n = 5} \quad \begin{cases} -1 a_5 = 4, \\ -1 b_5 = 0, \end{cases}$$

$$\boxed{n \in \mathbb{N} \setminus \{0, 1, 2, 5\}} \quad \begin{cases} (1 + (-1)^n)a_n = 0, \\ (1 + (-1)^n)b_n = 0. \end{cases}$$

Therefore, we conclude that

$$y(x) = -\cos(x) + \sin(2x) - 4\cos(5x).$$

#### Exercise 4.

We have

$$\begin{aligned} \hat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-x} e^{-i\alpha x} dx \\ &= \frac{1}{\sqrt{2\pi}} \lim_{N \rightarrow \infty} \int_0^N e^{-(1+i\alpha)x} dx \\ &= \frac{1}{\sqrt{2\pi}} \lim_{N \rightarrow \infty} \left[ \frac{-1}{1+i\alpha} e^{-(1+i\alpha)x} \right]_0^N \\ &= \frac{1}{\sqrt{2\pi}} \left( \frac{1}{1+i\alpha} - \frac{1}{1+i\alpha} \lim_{N \rightarrow \infty} e^{-(1+i\alpha)N} \right). \end{aligned}$$

Now,

$$\lim_{N \rightarrow \infty} \left| e^{-(1+i\alpha)N} \right| = \lim_{N \rightarrow \infty} e^{\operatorname{Re}(-(1+i\alpha)N)} = \lim_{N \rightarrow \infty} e^{-N} = 0,$$

and therefore  $\lim_{N \rightarrow \infty} e^{-(1+i\alpha)N} = 0$ .

Alternatively,

$$\lim_{N \rightarrow \infty} \operatorname{Re} \left( e^{-(1+i\alpha)N} \right) = \lim_{N \rightarrow \infty} e^{-N} \cos(\alpha N) = 0,$$

$$\lim_{N \rightarrow \infty} \operatorname{Im} \left( e^{-(1+i\alpha)N} \right) = \lim_{N \rightarrow \infty} e^{-N} \sin(\alpha N) = 0.$$

Finally,

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{1+i\alpha}.$$

**Exercise 5.**

The function  $f$  is piecewise-defined and we can verify that:

$$\int_{-\infty}^{+\infty} |f(x)| dx = \int_{-1}^1 1 dx = 2 < +\infty.$$

Thus, the Fourier transform is well-defined. We compute:

$$\begin{aligned} \hat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx = \int_{-1}^1 e^{-i\alpha x} dx \\ &= -\frac{1}{\sqrt{2\pi}} \left[ \frac{e^{-i\alpha x}}{i\alpha} \right]_{-1}^1 = \frac{1}{\sqrt{2\pi}} \frac{e^{i\alpha} - e^{-i\alpha}}{i\alpha} = \sqrt{\frac{2}{\pi}} \frac{\sin \alpha}{\alpha}. \end{aligned}$$

**Exercise 6.**

- a) The function  $f$  is piecewise-defined and integrable over  $\mathbb{R}$ . Its Fourier transform is

$$\begin{aligned} \hat{f}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 \sqrt{\frac{\pi}{2}} e^{-i\alpha x} dx \\ &= \frac{1}{2} \left[ \frac{1}{-i\alpha} e^{-i\alpha x} \right]_{-1}^1 = \frac{1}{2i\alpha} (e^{i\alpha} - e^{-i\alpha}) = \frac{\sin(\alpha)}{\alpha}. \end{aligned}$$

- b) The function  $g$  is continuous and integrable over  $\mathbb{R}$ . Its Fourier transform

is

$$\begin{aligned}
\hat{g}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x) e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-2}^0 \left(\pi + \frac{\pi}{2}x\right) e^{-i\alpha x} dx + \frac{1}{\sqrt{2\pi}} \int_0^2 \left(\pi - \frac{\pi}{2}x\right) e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{-i\alpha} e^{-i\alpha x} \left(\pi + \frac{\pi}{2}x\right) \right]_{-2}^0 - \frac{1}{\sqrt{2\pi}} \int_{-2}^0 \frac{\pi}{-2i\alpha} e^{-i\alpha x} dx \\
&\quad + \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{-i\alpha} e^{-i\alpha x} \left(\pi - \frac{\pi}{2}x\right) \right]_0^2 - \frac{1}{\sqrt{2\pi}} \int_0^2 \frac{\pi}{2i\alpha} e^{-i\alpha x} dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{\pi}{2} \left( \left[ \frac{e^{-i\alpha x}}{-\alpha^2} \right]_0^2 - \left[ \frac{e^{-i\alpha x}}{-\alpha^2} \right]_{-2}^0 \right) \\
&= \frac{\sqrt{\pi}}{2\sqrt{2}} \left( \frac{e^{2i\alpha} + e^{-2i\alpha}}{2} \frac{2}{-\alpha^2} - \frac{2}{-\alpha^2} \right) \\
&= \frac{\sqrt{\pi}}{-\alpha^2 \sqrt{2}} (\cos(2\alpha) - 1) = \sqrt{2\pi} \frac{\sin^2(\alpha)}{\alpha^2}.
\end{aligned}$$

We observe that  $\hat{g}(\alpha) = \sqrt{2\pi} \hat{f}(\alpha)^2$ . So  $g = f * f$ .

### Exercise 7.

1. We first compute

$$\mathfrak{F}_c(f)(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \cos(\alpha x) dx.$$

Since  $f(x) = e^{-x} \cos x$ , we obtain

$$\begin{aligned}
\mathfrak{F}_c(f)(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} e^{-x} \left[ \frac{\cos((\alpha+1)x) + \cos((\alpha-1)x)}{2} \right] dx \\
&= \sqrt{\frac{2}{\pi}} \left[ -\frac{e^{-x} [\cos((\alpha-1)x) - (\alpha-1) \sin((\alpha-1)x)]}{2(1+(\alpha-1)^2)} \right. \\
&\quad \left. + \frac{e^{-x} [(\alpha+1) \sin((\alpha+1)x) - \cos((\alpha+1)x)]}{2(1+(\alpha+1)^2)} \right]_0^{+\infty}.
\end{aligned}$$

Therefore,

$$\mathfrak{F}_c(f)(\alpha) = \sqrt{\frac{2}{\pi}} \frac{2 + \alpha^2}{(2 - 2\alpha + \alpha^2)(2 + 2\alpha + \alpha^2)}.$$

2. We now compute

$$\mathfrak{F}_s(f)(\alpha) = -i\sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(x) \sin(\alpha x) dx.$$

Again using  $f(x) = e^{-x} \cos x$ , we obtain

$$\begin{aligned} \mathfrak{F}_s(f)(\alpha) &= -i\sqrt{\frac{2}{\pi}} \int_0^{+\infty} e^{-x} \left[ \frac{\sin((\alpha+1)x) + \sin((\alpha-1)x)}{2} \right] dx \\ &= -i\sqrt{\frac{2}{\pi}} \left[ \frac{e^{-x} [-(\alpha-1) \cos((\alpha-1)x) - \sin((\alpha-1)x)]}{2(1+(\alpha-1)^2)} \right. \\ &\quad \left. - \frac{e^{-x} [(\alpha+1) \cos((\alpha+1)x) + \sin((\alpha+1)x)]}{2(1+(\alpha+1)^2)} \right]_0^{+\infty}. \end{aligned}$$

Thus,

$$\mathfrak{F}_s(f)(\alpha) = -i\sqrt{\frac{2}{\pi}} \frac{\alpha^3}{(2-2\alpha+\alpha^2)(2+2\alpha+\alpha^2)}.$$