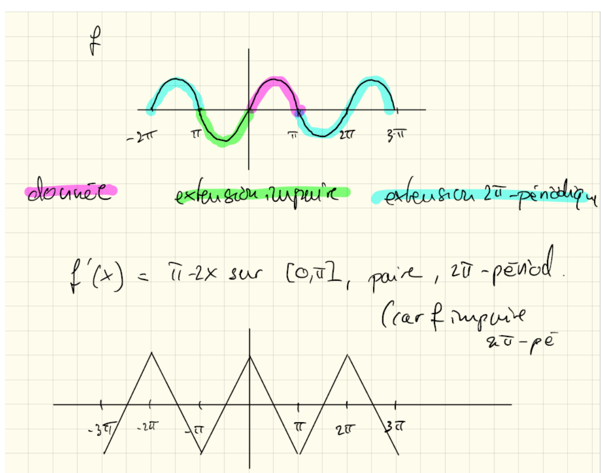
EPFL - Fall 2025	Dr. Pablo Antolin
Analysis III – MATH-202(c) – EL GM MX	**Exercises**
Correction 11	November 25 2025

Exercise 1. 1. Since f is odd, $a_n = 0$ for all n. Moreover,

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x(\pi - x)\sin(nx)\,dx && \Big|\ f \text{ odd implies } f(x)\sin(nx) \text{ even} \\
&\overset{\text{IBP}}{=} -\frac{2}{\pi n}\left[x(\pi - x)\cos(nx)\right]_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}(\pi - 2x)\cos(nx)\,dx && \Big|\ u = x(\pi - x) \quad v = -\tfrac{1}{n}\cos(nx),\ u' = \pi - 2x \quad v' = \sin(nx) \\
&\overset{\text{IBP}}{=} \frac{2}{\pi n^2}\left[(\pi - 2x)\sin(nx)\right]_0^{\pi} + \frac{4}{\pi n^2}\int_0^{\pi}\sin(nx)\,dx && \Big|\ u = \pi - 2x \quad v = \tfrac{1}{n}\sin(nx),\ u' = -2 \quad v' = \cos(nx) \\
&= \frac{4}{\pi n^3}\left[-\cos(nx)\right]_0^{\pi} = \frac{4}{\pi n^3}\left(-\cos(n\pi) + 1\right) \\
&= \frac{4}{\pi n^3}\left((-1)^{n+1} + 1\right) \\
&= \begin{cases} 0 & \text{if } n \text{ is even,} \\ \dfrac{8}{\pi(2k-1)^3} & \text{if } n = 2k - 1 \text{ is odd.} \end{cases}
\end{aligned}
$$

Thus,

$$Ff(x) = \sum_{k=1}^{\infty} \frac{8}{\pi(2k-1)^3}\sin((2k-1)x).$$



2. From the graph, we see that f is piecewise C^1, so by Parseval:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{k=1}^{\infty}\frac{64}{\pi^2}\frac{1}{(2k-1)^6}.$$

Since f^2 is even,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi}(x^4 - 2\pi x^3 + \pi^2 x^2)\,dx = \frac{\pi^4}{15}.$$

Thus,

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^6} = \frac{\pi^6}{960}.$$

Exercise 2.
Remark: The purpose of this exercise is not to compute an antiderivative of $\cos^4(x)$ (which can be done using Euler's formulas), but rather to compute a Fourier series using Euler's formulas.

Here $f(x) = \cos^2(x)$, so that

$$\int_{-\pi}^{\pi}\cos^4(x)\,dx = \int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(x)\,dx.$$

$$\cos^2(x) = \frac{e^{2ix} + 2 + e^{-2ix}}{4} = \frac{1}{2} + \frac{1}{2}\cos(2x),$$

so the Fourier coefficients of $f(x) = \cos^2(x)$ are $a_0 = 1$, $a_2 = \frac{1}{2}$, and all others vanish. Therefore Parseval gives

$$\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2),$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^4(x)\,dx = \frac{1}{2} + \frac{1}{4} = \frac{3}{4},$$

$$\Rightarrow \int_{-\pi}^{\pi}\cos^4(x)\,dx = \frac{3}{4}\pi,$$

which is the desired result.

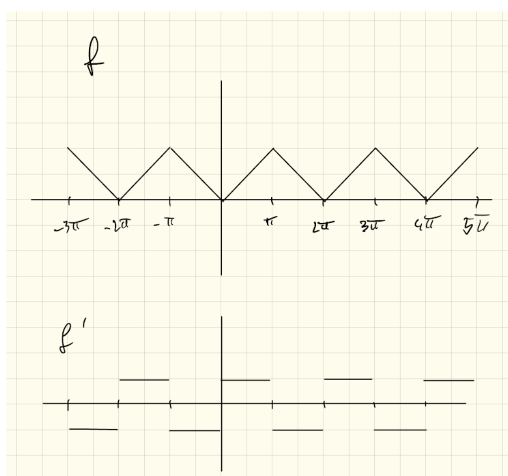
Exercise 3. 1. Since f is even, we have $b_n = 0$ for all n. Moreover,

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\, dx = \frac{2}{\pi}\int_0^{\pi} x\, dx = \pi, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(nx)\, dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\, dx \\
&\overset{\text{IBP}}{=} \frac{2}{\pi n}\left[x\sin(nx)\right]_0^{\pi} - \frac{2}{\pi n}\int_0^{\pi} \sin(nx)\, dx \qquad \left|\ \begin{array}{ll} u = x & v = \frac{1}{n}\sin(nx) \\ u' = 1 & v' = \cos(nx) \end{array}\right. \\
&= \frac{2}{\pi n^2}\left[\cos(nx)\right]_0^{\pi} = \frac{2}{\pi n^2}\left((-1)^n - 1\right) \\
&= \begin{cases} 0 & \text{if } n \text{ is even,} \\ -\dfrac{4}{\pi(2k+1)^2} & \text{if } n = 2k+1 \text{ is odd.} \end{cases}
\end{aligned}
$$

Thus,

$$
Ff(x) = \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{-4}{\pi(2k+1)^2}\cos((2k+1)x).
$$

2. Note that the Fourier coefficients behave like $\frac{1}{k^2}$, whereas the series we want behaves like $\frac{1}{k^4}$. Since the exponent in the series is twice that of the Fourier coefficients, we use Parseval.



From the graph, f is piecewise C^1, so

$$
\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{\pi^2}{2} + \sum_{k=0}^{\infty} \frac{16}{\pi^2}\frac{1}{(2k+1)^4}.
$$

On the other hand,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2\pi^2}{3}.$$

Thus,

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{\pi^2}{16}\left(\frac{2\pi^2}{3} - \frac{\pi^2}{2}\right) = \frac{\pi^4}{96}.$$

Furthermore,

$$S = \sum_{n=1}^{\infty}\frac{1}{n^4} = \sum_{k=1}^{\infty}\frac{1}{(2k)^4} + \sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{1}{16}\sum_{k=1}^{\infty}\frac{1}{k^4} + \frac{\pi^4}{96} = \frac{1}{16}S + \frac{\pi^4}{96}.$$

This yields

$$S = \frac{16}{15}\cdot\frac{\pi^4}{96} = \frac{\pi^4}{90}.$$

Exercise 4.

1. On one hand we have, $a_0 = \frac{2}{\pi}\int_0^{\pi} x\,\mathrm{d}x = \pi$ and on the other, for $n \in \mathbb{N}^*$, integrating by parts we obtain,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\left(\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\Big|_{x=0}^{x=\pi}\right) = \frac{2\left((-1)^n - 1\right)}{\pi n^2};$$
$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\left(-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\Big|_{x=0}^{x=\pi}\right) = -\frac{2\,(-1)^n}{n}.$$

Thus, the Fourier cosine series of f is given by

$$F_c f\,(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2}\,\cos\left((2n+1)\,x\right),$$

and the Fourier sine series of f is given by

$$F_s f\,(x) = -2\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}\,\sin(nx).$$

2. The Fourier cosine series $F_c(f)$ corresponds to the Fourier series of a function $g : \mathbb{R} \to \mathbb{R}$ that is 2π-periodic and *even*, and equal to f over $[0, \pi[$.

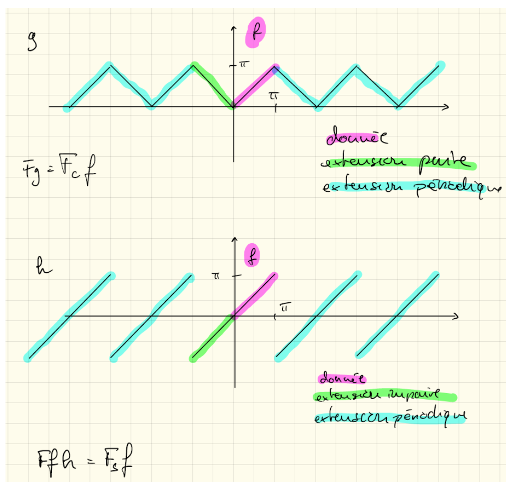
As the function g is continuous over $[0, \pi]$, applying Dirichlet's theorem we have

$$F_c f(x) = Fg(x) = g(x) = f(x), \qquad \text{for every } x \in [0, \pi].$$

The Fourier sine serie $F_s(f)$ corresponds to the Fourier series of a function $h : \mathbb{R} \to \mathbb{R}$ that is 2π-periodic and *odd*, and equal to f over $[0, \pi[$. The function h is continuous over $[0, \pi[$ and has a discontinuity in π. Thus, applying Dirichlet's theorem

$$F_s f(x) = Fh(x) = h(x) = f(x), \quad \text{pour tout } x \in [0, \pi[, \text{ and}$$

$$F_s f(\pi) = Fh(\pi) = \frac{h(\pi - 0) + h(\pi + 0)}{2} = 0 \neq \pi = f(\pi).$$



3. After the previous point, we find

$$\frac{\pi}{2} = f\left(\frac{\pi}{2}\right) = F_s f\left(\frac{\pi}{2}\right) = 2\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1},$$

and then $\displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$. In the same way, we have

$$0 = f(0) = F_c f(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}.$$

Thus, $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

4. Applying Parseval's identity to $F_c f = Fg$, we find,

$$\frac{a_0^2}{2} + \sum_{k=1}^{+\infty} a_k^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2} \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4} = \frac{2}{2\pi} \int_{-\pi}^{\pi} |x|^2 \mathrm{d}x = \frac{2\pi^2}{3}.$$

Thus, $\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$.

Exercise 5.

1. As the function f is 2π-periodic and odd, we conclude that

$$\int_0^{2\pi} f(y)\,dy = \int_{-\pi}^{\pi} f(y)\,dy = 0,$$

and then F is 2π-periodic. We can also show that F is odd. Indeed,

$$F(-x) = \int_0^{-x} f(y)\,dy = -\int_0^{x} f(-z)\,dz = \int_0^{x} f(z)\,dz = F(x),$$

where we used the change of variable $z = -y$, and the fact that f is odd, In addition,

$$F(x) = \int_0^x y(\pi - y)\,dy = \pi\frac{x^2}{2} - \frac{x^3}{3}, \quad x \in [0, \pi].$$

2. Using the Theorem 14.6 of the book (check also Theorem 2 in section 4.3.3 of the course notes) about integration of Fourier series.

$$\begin{aligned} F(x) = \int_0^x f(y)\,dy &= \int_0^x \frac{a_0^f}{2}\,dy + \sum_{n=1}^{\infty} \int_0^x \left[a_n^f \cos(ny) + b_n^f \sin(ny)\right]\,dy \\ &= \frac{a_0^f}{2}x + \sum_{n=1}^{\infty}\left[\frac{a_n^f}{n}\sin(nx) - \frac{b_n^f}{n}\cos(nx) + \frac{b_n^f}{n}\right] \\ &= \frac{8}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} - \frac{8}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}\cos\left((2n+1)x\right), \end{aligned}$$

since the Fourier coefficients of the function f are given by

$$a_n^f = 0, \quad b_{2n}^f = 0, \quad \text{and} \quad b_{2n+1}^f = \frac{8}{\pi(2n+1)^3}, \quad \text{for every } n \geq 0.$$

For finding the constant term, we proceed int he following way: because F is even

$$\begin{aligned} a_0^F &= \frac{2}{2\pi}\int_{-\pi}^{\pi} F(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} F(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}\left[\pi\frac{x^2}{2} - \frac{x^3}{3}\right]\mathrm{d}x \\ &= \frac{2}{\pi}\left(\frac{\pi^4}{6} - \frac{\pi^4}{12}\right) = \frac{\pi^3}{6}. \end{aligned}$$

Thus, the Fourier series of F is given by

$$\frac{\pi^3}{12} - \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \cos\left((2n+1)x\right).$$

Remark: Using the point 4 of Execise 4, we obtain

$$\frac{8}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^3}{12},$$

and conclude that $a_0^F = \pi^3/6$.