EPFL - Fall 2025	Dr. Pablo Antolin
Analysis III – MATH-202(c) – EL GM MX	**Exercises**
Correction 10	November 18 2025

Exercise 1.

1. We find after computation

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} e^{(x-\pi)}\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{y}\cos(ny+n\pi)\,dy \\ &= \frac{(-1)^n}{\pi}\int_{-\pi}^{\pi} e^{y}\cos(ny)\,dy = \frac{2\sinh\pi}{(1+n^2)\,\pi}, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} e^{(x-\pi)}\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{y}\sin(ny+n\pi)\,dy \\ &= \frac{(-1)^n}{\pi}\int_{-\pi}^{\pi} e^{y}\sin(ny)\,dy = \frac{-2n\sinh\pi}{(1+n^2)\,\pi}. \end{aligned}$$

2. Thanks to Dirichlet's theorem, we have that if

$$Ff(x) = \frac{\sinh\pi}{\pi} + \frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}\left(\frac{2\cos(nx)}{1+n^2} - \frac{2n\sin(nx)}{1+n^2}\right),$$

then

$$Ff(x) = f(x) \qquad \text{if } x\in(0,2\pi).$$

3. In particular, if $x=\pi$ we obtain

$$1 = \frac{\sinh\pi}{\pi}\left(1+\sum_{n=1}^{\infty}\frac{2(-1)^n}{1+n^2}\right),$$

from which

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2} = \frac{\pi}{2\sinh\pi} - \frac{1}{2},$$

that is,

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2} = \frac{\pi}{2\sinh\pi} = \frac{\pi}{e^{\pi}-e^{-\pi}}.$$

Exercise 2.

1. We have

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi}(x-\pi)^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} y^2\,dy = \frac{2}{3}\pi^2, \\ a_n &= \frac{1}{\pi}\int_0^{2\pi}(x-\pi)^2\cos(nx)\,dx = \frac{(-1)^n}{\pi}\int_{-\pi}^{\pi} y^2\cos(ny)\,dy = \frac{4}{n^2}, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi}(x-\pi)^2\sin(nx)\,dx = \frac{(-1)^n}{\pi}\int_{-\pi}^{\pi} y^2\sin(ny)\,dy = 0. \end{aligned}$$

   The Fourier series is therefore

$$Ff(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4\cos(nx)}{n^2}.$$

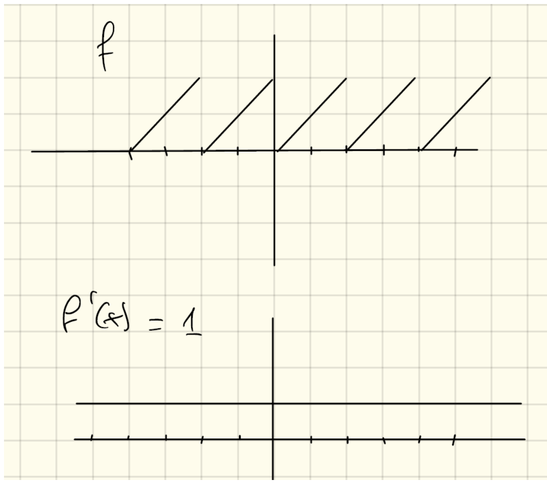
2. Dirichlet's theorem ensures (note that f is continuous at $x = 0$ and $x = 2\pi$) that

$$Ff(x) = f(x) \qquad \text{for } x \in [0, 2\pi].$$

3. Taking $x = 0$ (since $f(0) = \pi^2$) and $x = \pi$ (since $f(\pi) = 0$), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{4}{n^2} &= \frac{2\pi^2}{3} \quad &\Rightarrow \quad \sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{\pi^2}{6}, \\ \sum_{n=1}^{\infty}4\frac{(-1)^n}{n^2} &= -\frac{\pi^2}{3} \quad &\Rightarrow \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} &= -\frac{\pi^2}{12}. \end{aligned}$$

Exercise 3.



Here we have $T = 2$ and $f(x) = x$ for $x \in (0, 2)$. Thus, for $n \in \mathbb{Z}^*$,

$$\begin{aligned} c_0 &= \frac{1}{T} \int_0^T f(x)\, dx \\ &= \frac{1}{2} \int_0^2 x\, dx \\ &= \frac{1}{2} \left[\frac{1}{2} x^2 \right]_{x=0}^{x=2} = 1 \end{aligned}$$

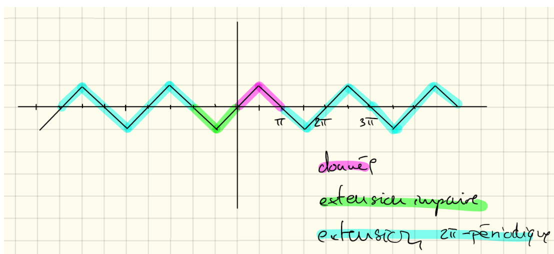
$$\begin{aligned} c_n &= \frac{1}{T} \int_0^T f(x) e^{-i\frac{2\pi n}{T}x}\, dx \\ &= \frac{1}{2} \int_0^2 \underbrace{x}_{u} \underbrace{e^{-in\pi x}}_{v'} dx \qquad \left| \begin{array}{ll} u = x & v = -\frac{1}{in\pi} e^{-in\pi x} \\ u' = 1 & v' = e^{-in\pi x} \end{array} \right. \\ &\overset{\text{IBP}}{=} \frac{1}{2} \left[-\frac{x}{in\pi} e^{-in\pi x} \right]_{x=0}^{x=2} + \frac{1}{2in\pi} \int_0^2 e^{-in\pi x}\, dx \\ &= -\frac{1}{in\pi} e^{-in2\pi} + \frac{1}{2in\pi} \left[\frac{-1}{in\pi} e^{-in\pi x} \right]_{x=0}^{x=2} \\ &= -\frac{1}{in\pi} - \frac{1}{2i^2n^2\pi^2} e^{-in2\pi} + \frac{1}{2i^2n^2\pi^2} \\ &= \frac{i}{n\pi} \end{aligned}$$

Therefore,

$$Ff(x) = \sum_{n\in\mathbb{Z}} c_n e^{i\frac{2\pi n}{T}x} = 1 + \sum_{n\in\mathbb{Z}^*} \frac{i}{n\pi} e^{in\pi x}$$

Exercise 4.
The odd extension of f to $]-\pi, \pi]$ is given by

$$f(x) = \begin{cases} f(x) & \text{if } 0 \le x \le \pi, \\ -f(-x) & \text{if } -\pi < x < 0 \end{cases} = \begin{cases} -x-\pi & \text{if } -\pi < x < \frac{\pi}{2}, \\ x & \text{if } -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \\ \pi - x & \text{if } \frac{\pi}{2} < x \le \pi \end{cases}$$



Variant 1: Definition of the complex coefficients.

We have $T = 2\pi$, and we know f on $]-\pi, \pi]$. Thus,

$$\begin{aligned} c_n &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) e^{-i\frac{2\pi n}{T}x}\, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \\ &= \frac{1}{2\pi} \left(\int_{-\pi}^{-\pi/2} (-x-\pi) e^{-inx}\, dx + \int_{-\pi/2}^{\pi/2} x e^{-inx}\, dx + \int_{\pi/2}^{\pi} (\pi - x) e^{-inx}\, dx \right) \end{aligned}$$

Note that, by periodicity, one could combine the first and the last integrals into a single one:

$$\int_{-\pi}^{-\pi/2} (-x-\pi) e^{-inx}\, dx + \int_{-\pi/2}^{\pi} (\pi - x) e^{-inx}\, dx = \int_{\pi/2}^{3\pi/2} (\pi - x) e^{-inx}\, dx.$$

In short, we replace the green descending branch in the graph with the blue descending branch just to the right of the purple part.

However, since the task waiting for me after writing this solution is particularly unpleasant, I'd rather write a longer solution and compute all three integrals separately.

$$
\begin{aligned}
I_1 &= \int_{-\pi}^{-\pi/2} (-x-\pi)e^{-inx}\,dx \\
&\overset{\text{IBP}}{=} \left[(x+\pi)\frac{1}{in}e^{-inx}\right]_{-\pi}^{-\pi/2} - \frac{1}{in}\int_{-\pi}^{-\pi/2} e^{-inx}\,dx \qquad \left|\begin{array}{ll} u=-x-\pi & v=\frac{-1}{in}e^{-inx} \\ u'=-1 & v'=e^{-inx}\end{array}\right. \\
&= \frac{\pi}{2}\frac{1}{in}e^{in\frac{\pi}{2}} - \frac{1}{n^2}\left[e^{-inx}\right]_{-\pi}^{-\pi/2} \\
&\overset{i^{-1}=-i}{=} -\frac{i\pi}{2n}e^{in\frac{\pi}{2}} - \frac{1}{n^2}e^{in\frac{\pi}{2}} + \frac{1}{n^2}(-1)^n.
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \int_{-\pi/2}^{\pi/2} xe^{-inx}\,dx \\
&= \frac{-1}{in}\left[xe^{-inx}\right]_{-\pi/2}^{\pi/2} + \frac{1}{in}\int_{-\pi/2}^{\pi/2} e^{-inx}\,dx \qquad \left|\begin{array}{ll} u=x & v=\frac{-1}{in}e^{-inx} \\ u'=1 & v'=e^{-inx}\end{array}\right. \\
&= -\frac{\pi}{2in}e^{-in\frac{\pi}{2}} - \frac{\pi}{2in}e^{in\frac{\pi}{2}} + \frac{1}{n^2}\left[e^{-inx}\right]_{-\pi/2}^{\pi/2} \\
&\overset{i^{-1}=-i}{=} \frac{i\pi}{2n}e^{-in\frac{\pi}{2}} + \frac{i\pi}{2n}e^{in\frac{\pi}{2}} + \frac{1}{n^2}e^{-in\frac{\pi}{2}} - \frac{1}{n^2}e^{in\frac{\pi}{2}}.
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \int_{\pi/2}^{\pi} (\pi-x)e^{-inx}\,dx \\
&\overset{\text{IBP}}{=} \frac{i}{n}\left[(\pi-x)e^{-inx}\right]_{\pi/2}^{\pi} + \frac{i}{n}\int_{\pi/2}^{\pi} e^{-inx}\,dx \qquad \left|\begin{array}{ll} u=\pi-x & v=\frac{i}{n}e^{-inx} \\ u'=-1 & v'=e^{-inx}\end{array}\right. \\
&= -\frac{i\pi}{2n}e^{-in\frac{\pi}{2}} - \frac{1}{n^2}\left[e^{-inx}\right]_{\pi/2}^{\pi} \\
&= -\frac{i\pi}{2n}e^{-in\frac{\pi}{2}} + \frac{1}{n^2}e^{-in\frac{\pi}{2}} - \frac{1}{n^2}(-1)^n.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I_1+I_2+I_3 &= -\frac{2}{n^2}e^{in\frac{\pi}{2}} + \frac{2}{n^2}e^{-in\frac{\pi}{2}} = -\frac{4i}{n^2}\sin\left(n\frac{\pi}{2}\right) \\
&= \begin{cases} 0 & \text{if } n=2k,\ k\in\mathbb{Z}, \\ -\dfrac{4i}{(2k-1)^2}\sin\left(k\pi-\dfrac{\pi}{2}\right) & \text{if } n=2k-1,\ k\in\mathbb{Z} \end{cases} \\
&= \begin{cases} 0 & \text{if } n=2k,\ k\in\mathbb{Z}, \\ \dfrac{4i(-1)^k}{(2k-1)^2} & \text{if } n=2k-1,\ k\in\mathbb{Z}. \end{cases}
\end{aligned}
$$

This gives

$$c_n = \frac{1}{2\pi}(I_1 + I_2 + I_3) = \begin{cases} 0 & \text{if } n = 2k,\ k \in \mathbb{Z}, \\ \dfrac{2i(-1)^k}{\pi(2k-1)^2} & \text{if } n = 2k-1,\ k \in \mathbb{Z}. \end{cases}$$

Variant 2: Compute the real coefficients and then use the formulas to obtain the complex ones.

Since f is odd, all the cosine Fourier coefficients vanish. Moreover, for $n \geq 1$,

$$\begin{aligned} b_n &= \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\sin\left(\frac{2\pi n}{T}x\right)\,dx \\ &= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} x\sin(nx)\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\sin(nx)\,dx\right) \end{aligned}$$

We compute an antiderivative of $x\sin(ax)$ for $a \in \mathbb{R}$:

$$\begin{aligned} \int^x t\sin(at)\,dt &\overset{\text{IBP}}{=} \left[-\frac{t}{a}\cos(at)\right]^{t=x} + \int^x \frac{1}{a}\cos(at)\,dt \qquad \left|\ \begin{matrix} u = t & v = -\frac{1}{a}\cos(at) \\ u' = 1 & v' = \sin(at) \end{matrix}\right. \\ &= -\frac{x}{a}\cos(ax) + \frac{1}{a^2}\sin(ax) \end{aligned}$$

Thus,

$$\begin{aligned} b_n =& \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin(nx)\,dx - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} x\sin(nx)\,dx + 2\int_{\frac{\pi}{2}}^{\pi}\sin(nx)\,dx \\ =& \frac{2}{\pi}\left[\frac{1}{n^2}\sin(nx) - \frac{x}{n}\cos(nx)\right]_{x=0}^{x=\frac{\pi}{2}} - \frac{2}{\pi}\left[\frac{1}{n^2}\sin(nx) - \frac{x}{n}\cos(nx)\right]_{x=\frac{\pi}{2}}^{x=\pi} + 2\left[\frac{-1}{n}\cos(nx)\right]_{x=\frac{\pi}{2}}^{x=\pi} \\ =& \frac{2}{n^2\pi}\sin\left(n\frac{\pi}{2}\right) - \frac{1}{n}\cos\left(n\frac{\pi}{2}\right) - \frac{2}{n^2\pi}\underbrace{\sin(n\pi)}_{=0} + \frac{2}{n}\cos(n\pi) + \frac{2}{n^2\pi}\sin\left(n\frac{\pi}{2}\right) \\ & - \frac{1}{n}\cos\left(n\frac{\pi}{2}\right) - \frac{2}{n}\cos(n\pi) + \frac{2}{n}\cos\left(n\frac{\pi}{2}\right) \\ =& \frac{4}{n^2\pi}\sin\left(n\frac{\pi}{2}\right) \end{aligned}$$

Finally,

$$\text{for } n \geq 1, \quad c_n = \frac{a_n - ib_n}{2} = \frac{-i}{2}b_n = \frac{-2i}{n^2\pi}\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & \text{if } n = 2k, \\ \dfrac{2i(-1)^k}{(2k-1)^2\pi} & \text{if } n = 2k-1, \end{cases}$$

$$\text{for } n \leq -1, \quad c_n = \frac{a_{-n} + ib_{-n}}{2} = \frac{i}{2}b_{-n} = \frac{2i}{n^2\pi}\sin\left(-n\frac{\pi}{2}\right) = \begin{cases} 0 & \text{if } n = 2k, \\ \dfrac{2i(-1)^k}{(2k-1)^2\pi} & \text{if } n = 2k-1. \end{cases}$$

We conclude that

$$Ff(x) = \sum_{k\in\mathbb{Z}} \frac{2i(-1)^k}{\pi(2k-1)^2} e^{i(2k-1)x}.$$

From the graph, we observe that f is piecewise C^1 and continuous. Hence, for all x,

$$Ff(x) \overset{\text{(Dirichlet)}}{=} \frac{f(x-0)+f(x+0)}{2} \overset{f\in C^0}{=} f(x).$$

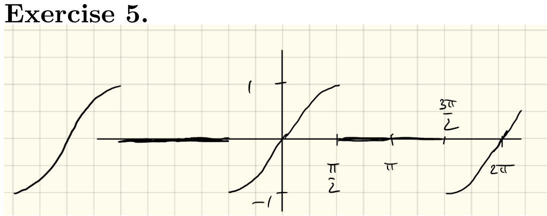
To compute $\sum_{k\in\mathbb{Z}} \frac{1}{(2k-1)^2}$ from $\sum_{k\in\mathbb{Z}} \frac{2i(-1)^k}{\pi(2k-1)^2} e^{i(2k-1)x}$, we want to choose x such that $e^{i(2k-1)x} \approx i(-1)^k$. Here, $x = \pm\frac{\pi}{2}$ works. Then,

$$\frac{\pi}{2} = f\left(\frac{\pi}{2}\right) = Ff\left(\frac{\pi}{2}\right) = \sum_{k\in\mathbb{Z}} \frac{2i(-1)^k}{\pi(2k-1)^2} \underbrace{e^{ik\pi}}_{(-1)^k} \underbrace{e^{-i\frac{\pi}{2}}}_{-i} = \frac{2}{\pi}\sum_{k\in\mathbb{Z}} \frac{1}{(2k-1)^2},$$

and therefore

$$\sum_{k\in\mathbb{Z}} \frac{1}{(2k-1)^2} = \frac{\pi^2}{4}.$$

Exercise 5.



Since f is odd, we have $a_n = 0$ for all n, and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{2\pi}\left(\int_0^{\pi/2} \sin(x)\sin(nx)\,dx + \int_{3\pi/2}^{2\pi} \sin(x)\sin(nx)\,dx\right).$$

We can compute the primitives of the integrands using Euler's formulas:

$$\begin{aligned}
\sin(x)\sin(nx) &= \frac{e^{ix}-e^{-ix}}{2i}\cdot\frac{e^{inx}-e^{-inx}}{2i} \\
&= -\frac{1}{4}\left(e^{i(n+1)x} - e^{-i(n-1)x} - e^{i(n-1)x} + e^{-i(n+1)x}\right) \\
&= \frac{1}{2}\left(\cos((n-1)x) - \cos((n+1)x)\right),
\end{aligned}$$

and thus

$$\int \sin(x)\sin(nx)\,dx = \begin{cases} \frac{1}{2}\left(\frac{\sin((n-1)x)}{n-1} - \frac{\sin((n+1)x)}{n+1}\right), & \text{if } n \neq 1, \\ \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right), & \text{if } n = 1. \end{cases}$$

We now compute the integrals using integration by parts for $n \neq 1$.

Case $n = 1$:

$$I_1 = \int_0^{\pi/2} \sin^2(x)\,dx = \frac{1}{2}\left[x - \cos(x)\sin(x)\right]_0^{\pi/2} = \frac{\pi}{4},$$
$$I_2 = \int_{3\pi/2}^{2\pi} \sin^2(x)\,dx = \frac{1}{2}\left[x - \cos(x)\sin(x)\right]_{3\pi/2}^{2\pi} = \pi - \frac{3\pi}{4} = \frac{\pi}{4}.$$

Hence,

$$b_1 = \frac{1}{\pi}(I_1 + I_2) = \frac{1}{2}.$$

Case $n > 1$:

$$
\begin{aligned}
I_1 &= \int_0^{\pi/2} \sin(x)\sin(nx)\,dx \\
&\overset{\text{IBP}}{=} \left[-\cos(x)\sin(nx)\right]_0^{\pi/2} + n\int_0^{\pi/2} \cos(x)\cos(nx)\,dx && \left|\ \begin{matrix} u = \sin(nx) & v = -\cos(x) \\ u' = n\cos(nx) & v' = \sin(x) \end{matrix}\right. \\
&\overset{\text{IBP}}{=} n\left[\sin(x)\cos(nx)\right]_0^{\pi/2} + n^2\int_0^{\pi/2} \sin(x)\sin(nx)\,dx && \left|\ \begin{matrix} u = \cos(nx) & v = \sin(x) \\ u' = -n\sin(nx) & v' = \cos(x) \end{matrix}\right. \\
&= n\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}\cos\left(n\frac{\pi}{2}\right) + n^2 I_1, \\
&\Rightarrow I_1 = n\cos\left(n\frac{\pi}{2}\right) + n^2 I_1.
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \int_{3\pi/2}^{2\pi} \sin(x)\sin(nx)\,dx \\
&\overset{\text{IBP}}{=} \left[-\cos(x)\sin(nx)\right]_{3\pi/2}^{2\pi} + n\int_{3\pi/2}^{2\pi} \cos(x)\cos(nx)\,dx && \left|\ \begin{matrix} u = \sin(nx) & v = -\cos(x) \\ u' = n\cos(nx) & v' = \sin(x) \end{matrix}\right. \\
&\overset{\text{IBP}}{=} n\left[\sin(x)\cos(nx)\right]_{3\pi/2}^{2\pi} + n^2\int_{3\pi/2}^{2\pi} \sin(x)\sin(nx)\,dx && \left|\ \begin{matrix} u = \cos(nx) & v = \sin(x) \\ u' = -n\sin(nx) & v' = \cos(x) \end{matrix}\right. \\
&= -n\underbrace{\sin\left(\frac{3\pi}{2}\right)}_{=-1}\cos\left(n\frac{3\pi}{2}\right) + n^2 I_2 \\
&= n\cos\left(n\frac{3\pi}{2}\right) + n^2 I_2.
\end{aligned}
$$

If n is even ($n = 2k$):

$$
\cos\left(n\frac{\pi}{2}\right) = \cos(k\pi) = (-1)^k, \quad \cos\left(n\frac{3\pi}{2}\right) = \cos(3k\pi) = (-1)^k,
$$

so

$$
\begin{aligned}
I_1 &= 2k(-1)^k + 4k^2 I_1 \quad \Rightarrow \quad I_1 = \frac{2k(-1)^{k+1}}{4k^2-1}, \\
I_2 &= 2k(-1)^k + 4k^2 I_2 \quad \Rightarrow \quad I_2 = \frac{2k(-1)^{k+1}}{4k^2-1},
\end{aligned}
$$

$$
I_1 + I_2 = \frac{4k(-1)^{k+1}}{4k^2-1}.
$$

If n is odd ($n = 2k-1$):

$$\cos\left(n\frac{\pi}{2}\right) = \cos\left(k\pi - \frac{\pi}{2}\right) = 0, \quad \cos\left(n\frac{3\pi}{2}\right) = \cos\left(3k\pi - \frac{3\pi}{2}\right) = 0,$$

and thus

$$I_1 = I_2 = 0 \quad \Rightarrow \quad I_1 + I_2 = 0.$$

Final result:

$$b_n = \frac{1}{\pi}(I_1 + I_2) = \begin{cases} \frac{1}{2}, & \text{if } n = 1, \\ \frac{4k(-1)^{k+1}}{\pi(4k^2-1)}, & \text{if } n = 2k \geq 2, \\ 0, & \text{if } n = 2k-1 \geq 3. \end{cases}$$

and therefore

$$Ff(x) = \frac{1}{2}\sin(x) + \sum_{k=1}^{\infty} \frac{4k(-1)^{k+1}}{\pi(4k^2-1)} \sin(2kx).$$

Exercise 6. 1. Since f is even, we have $b_n = 0$ for all n. Moreover,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} |\cos(x)|\,dx = \frac{1}{\pi}\left(\int_0^{\pi/2} \cos(x)\,dx - \int_{\pi/2}^{3\pi/2} \cos(x)\,dx + \int_{3\pi/2}^{2\pi} \cos(x)\,dx\right) = \frac{4}{\pi}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} |\cos(x)|\cos(nx)\,dx \\ &= \frac{1}{\pi}\left(\int_0^{\pi/2} \cos(x)\cos(nx)\,dx - \int_{\pi/2}^{3\pi/2} \cos(x)\cos(nx)\,dx + \int_{3\pi/2}^{2\pi} \cos(x)\cos(nx)\,dx\right) \end{aligned}$$

As in the previous exercise, we could compute these integrals by performing successive integrations by parts, leading to an equation involving the same integral. This time, however, we use Euler's formulas to directly obtain a primitive.

We have

$$\begin{aligned} \cos(x)\cos(nx) &= \frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^{inx} + e^{-inx}}{2} \\ &= \frac{1}{4}\left(e^{i(n+1)x} + e^{-i(n-1)x} + e^{i(n-1)x} + e^{-i(n+1)x}\right) \\ &= \frac{1}{2}\left(\cos((n+1)x) + \cos((n-1)x)\right). \end{aligned}$$

Hence,

$$\int \cos(x)\cos(nx)\,dx = \begin{cases} \dfrac{x}{2} + \dfrac{1}{4}\sin(2x), & \text{if } n = 1, \\[2ex] \dfrac{\sin((n+1)x)}{2(n+1)} + \dfrac{\sin((n-1)x)}{2(n-1)}, & \text{if } n \geq 2. \end{cases}$$

We then obtain, for $n = 1$,

$$\begin{aligned} \int_0^{2\pi} |\cos(x)|\cos(nx)\,dx = {} & \frac{1}{4}\sin(\pi) + \frac{\pi}{4} - 0 - 0 - \frac{1}{4}\sin(3\pi) - \frac{3\pi}{4} + \frac{1}{4}\sin(\pi) + \frac{\pi}{4} + \frac{1}{4}\sin(4\pi) \\ & + \pi - \frac{1}{4}\sin(3\pi) - \frac{3\pi}{2} = 0 \end{aligned}$$

and for $n \geq 2$,

$$\begin{aligned} \int_0^{2\pi} |\cos(x)|\cos(nx)\,dx = {} & \frac{1}{2(n+1)}\sin\Big((n+1)\frac{\pi}{2}\Big) + \frac{1}{2(n-1)}\sin\Big((n-1)\frac{\pi}{2}\Big) \\ & - \frac{1}{2(n+1)}\sin((n+1)0) + \frac{1}{2(n-1)}\sin((n-1)0) \\ & - \frac{1}{2(n+1)}\sin\left((n+1)\frac{3\pi}{2}\right) - \frac{1}{2(n-1)}\sin\left((n-1)\frac{3\pi}{2}\right) \\ & + \frac{1}{2(n+1)}\sin\Big((n+1)\frac{\pi}{2}\Big) + \frac{1}{2(n-1)}\sin\Big((n-1)\frac{\pi}{2}\Big) \\ & + \frac{1}{2(n+1)}\sin((n+1)2\pi) + \frac{1}{2(n-1)}\sin((n-1)2\pi) \\ & - \frac{1}{2(n+1)}\sin\left((n+1)\frac{3\pi}{2}\right) - \frac{1}{2(n-1)}\sin\left((n-1)\frac{3\pi}{2}\right) \\ = {} & \frac{1}{n+1}\sin\Big((n+1)\frac{\pi}{2}\Big) + \frac{1}{n-1}\sin\Big((n-1)\frac{\pi}{2}\Big) \\ & - \frac{1}{n+1}\sin\left((n+1)\frac{3\pi}{2}\right) - \frac{1}{n-1}\sin\left((n-1)\frac{3\pi}{2}\right) \end{aligned}$$

If n is odd, each term is zero. If n is even ($n = 2k$), then

$$\begin{aligned} \sin\Big((n+1)\frac{\pi}{2}\Big) &= \sin\Big(\frac{\pi}{2} + k\pi\Big) = (-1)^k, \\ \sin\Big((n-1)\frac{\pi}{2}\Big) &= \sin\Big(-\frac{\pi}{2} + k\pi\Big) = (-1)^{k+1}, \\ \sin\left((n+1)\frac{3\pi}{2}\right) &= \sin\left(\frac{3\pi}{2} + 3k\pi\right) = (-1)^{k+1}, \\ \sin\left((n-1)\frac{3\pi}{2}\right) &= \sin\left(-\frac{3\pi}{2} + 3k\pi\right) = (-1)^k, \end{aligned}$$

and therefore

$$\int_0^{2\pi} |\cos(x)|\cos(nx)\,dx = \frac{(-1)^k}{2k+1} + \frac{(-1)^{k+1}}{2k-1} - \frac{(-1)^{k+1}}{2k+1} - \frac{(-1)^k}{2k-1}$$
$$= \frac{2(-1)^k(2k-1-2k-1)}{4k^2-1} = \frac{4(-1)^{k+1}}{4k^2-1}.$$

Finally,

$$Ff(x) = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{\pi(4k^2-1)} \cos(2kx)$$

2. Observing the graph, we note that f is piecewise C^1 and continuous. Therefore, for all x,

$$Ff(x) \overset{\text{Dirichlet}}{=} \frac{f(x-0)+f(x+0)}{2} \overset{f\in C^0}{=} f(x)$$

To compute $\sum_{k=1}^{\infty} \frac{1}{4k^2-1}$ from $\sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{\pi(4k^2-1)} \cos(2kx)$, we must choose x such that $\cos(2kx) \approx (-1)^k$. Choosing $x = \frac{\pi}{2}$ works:

$$0 = |\cos(\tfrac{\pi}{2})| = f\left(\tfrac{\pi}{2}\right) = Ff\left(\tfrac{\pi}{2}\right) = \frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{\pi(4k^2-1)}(-1)^k = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{4k^2-1}$$

and hence

$$\sum_{k=1}^{\infty} \frac{1}{4k^2-1} = \frac{1}{2}.$$

Exercise 7. 1. Since f is even, the sine coefficients vanish. For the cosine coefficients, we have $T = 2\pi$, $f(x) = \cos(\alpha x)$ for $x \in (-\pi, \pi]$, and therefore

$$a_0 = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \cos(\alpha x)\,dx = \frac{2}{\alpha\pi}\left[\sin(\alpha x)\right]_{x=0}^{x=\pi} = \frac{2}{\alpha\pi}\sin(\alpha\pi)$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} \cos(\alpha x)\cos(nx)\,dx$$

Once again, we can choose between Euler's formulas or integration by parts to compute the integral:

Option 1: Using Euler's formulas.

$$\begin{aligned}\cos(\alpha x)\cos(nx) &= \frac{e^{i\alpha x}+e^{-i\alpha x}}{2}\cdot\frac{e^{inx}+e^{-inx}}{2}\\ &= \frac{1}{4}\left(e^{i(\alpha+n)x}+e^{i(\alpha-n)x}+e^{-i(\alpha-n)x}+e^{-i(\alpha+n)x}\right)\\ &= \frac{1}{2}\cos((\alpha+n)x)+\frac{1}{2}\cos((\alpha-n)x)\end{aligned}$$

$$\int^x \cos(\alpha x)\cos(nx)\,dx = \frac{1}{2(\alpha+n)}\sin((\alpha+n)x)+\frac{1}{2(\alpha-n)}\sin((\alpha-n)x).$$

Thus,

$$\begin{aligned}a_n &= \frac{2}{\pi}\left[\frac{1}{2(\alpha+n)}\sin((\alpha+n)x)+\frac{1}{2(\alpha-n)}\sin((\alpha-n)x)\right]_{x=0}^{x=\pi}\\ &= \frac{1}{\pi(\alpha+n)}\sin(\alpha\pi+n\pi)+\frac{1}{\pi(\alpha-n)}\sin(\alpha\pi-n\pi)\\ &= \frac{(-1)^n}{\pi(\alpha+n)}\sin(\alpha\pi)+\frac{(-1)^n}{\pi(\alpha-n)}\sin(\alpha\pi)\\ &= \frac{(-1)^n}{\pi}\frac{\alpha-n+\alpha+n}{\alpha^2-n^2}\sin(\alpha\pi)\\ &= \frac{2(-1)^n\alpha}{\pi(\alpha^2-n^2)}\sin(\alpha\pi)\end{aligned}$$

Option 2: Using Integration by Parts (IBP).

$$I = \int_0^\pi \underbrace{\cos(\alpha x)}_{=u'}\underbrace{\cos(nx)}_{v}\,dx \qquad \left|\ \begin{matrix} u=\frac{1}{\alpha}\sin(\alpha x) & v=\cos(nx)\\ u'=\cos(\alpha x) & v'=-n\sin(nx)\end{matrix}\right.$$

$$\begin{aligned}&\overset{\text{IBP}}{=} \left[\frac{1}{\alpha}\sin(\alpha x)\cos(nx)\right]_0^\pi+\frac{n}{\alpha}\int_0^\pi \sin(\alpha x)\sin(nx)\,dx\\ &= \frac{(-1)^n}{\alpha}\sin(\alpha\pi)+\frac{n}{\alpha}\int_0^\pi \underbrace{\sin(\alpha x)}_{=u'}\underbrace{\sin(nx)}_{v}\,dx \qquad \left|\ \begin{matrix} u=\frac{-1}{\alpha}\cos(\alpha x) & v=\sin(nx)\\ u'=\sin(\alpha x) & v'=n\cos(nx)\end{matrix}\right.\\ &\overset{\text{IBP}}{=} \frac{(-1)^n}{\alpha}\sin(\alpha\pi)-\frac{n}{\alpha}\left[\frac{1}{\alpha}\cos(\alpha x)\sin(nx)\right]_0^\pi+\frac{n^2}{\alpha^2}\int_0^\pi \cos(\alpha x)\cos(nx)\,dx\\ &= \frac{(-1)^n}{\alpha}\sin(\alpha\pi)+\frac{n^2}{\alpha^2}I\end{aligned}$$

Hence,

$$I = \frac{\frac{(-1)^n}{\alpha}\sin(\alpha\pi)}{1-\frac{n^2}{\alpha^2}} = \frac{\alpha(-1)^n}{\alpha^2-n^2}\sin(\alpha\pi),$$

and therefore,

$$a_n = \frac{2}{\pi} I = \frac{2(-1)^n \alpha}{\pi(\alpha^2 - n^2)} \sin(\alpha\pi).$$

Finally, the Fourier series of f is

$$\begin{aligned} Ff(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T} x\right) \\ &= \frac{1}{\alpha\pi} \sin(\alpha\pi) + \sum_{n=1}^{\infty} \frac{2(-1)^n \alpha}{\pi(\alpha^2 - n^2)} \sin(\alpha\pi) \cos(nx). \end{aligned}$$

2. To find the value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2}$$

from

$$\sum_{n=1}^{\infty} \frac{2(-1)^n \alpha}{\pi(\alpha^2 - n^2)} \sin(\alpha\pi) \cos(nx),$$

we must choose x such that $(-1)^n \cos(nx) \approx 1$. Choosing $x = \pi$ works, since $\cos(nx) = \cos(n\pi) = (-1)^n$, giving

$$Ff(\pi) = \frac{1}{\alpha\pi} \sin(\alpha\pi) - \frac{2\alpha \sin(\alpha\pi)}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2}.$$

Moreover, since f is continuous, we have $Ff(\pi) = f(\pi) = \cos(\alpha\pi)$.

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha \sin(\alpha\pi)} Ff(\pi) = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha \tan(\alpha\pi)},$$

which is the desired result.