

Exercise 1.

(i) Calculation of $\iint_{\Sigma} \operatorname{curl} F \cdot ds$. We have

$$\operatorname{curl} F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & z & x \end{vmatrix} = \begin{pmatrix} -1 \\ -1 \\ -x^2 \end{pmatrix}$$

and if $A = (0, 1) \times (0, 2\pi)$ then

$$\Sigma = \{\sigma(r, \theta) = (r^2 \cos \theta, r^2 \sin \theta, r) : (r, \theta) \in \bar{A}\}$$

The normal vector is given by

$$\sigma_r \wedge \sigma_\theta = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2r \cos \theta & 2r \sin \theta & 1 \\ -r^2 \sin \theta & r^2 \cos \theta & 0 \end{vmatrix} = \begin{pmatrix} -r^2 \cos \theta \\ -r^2 \sin \theta \\ 2r^3 \end{pmatrix}$$

and consequently

$$\begin{aligned} \iint_{\Sigma} \operatorname{curl} F \cdot ds &= \int_0^1 \int_0^{2\pi} (-1, -1, -r^4 \cos^2 \theta) \cdot (-r^2 \cos \theta, -r^2 \sin \theta, 2r^3) dr d\theta \\ &= - \int_0^1 \int_0^{2\pi} 2r^7 \cos^2 \theta dr d\theta = -\frac{\pi}{4} \end{aligned}$$

(ii) Calculation of $\int_{\partial \Sigma} F \cdot dl$. Moreover, we obtain that

$$\sigma(\partial A) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

where

$$\Gamma_1 = \{\sigma(r, 0) = (r^2, 0, r) : r : 0 \rightarrow 1\}$$

$$\Gamma_2 = \{\sigma(1, \theta) = (\cos \theta, \sin \theta, 1) : \theta : 0 \rightarrow 2\pi\}$$

$$\Gamma_3 = \{\sigma(r, 2\pi) = (r^2, 0, r) : r : 1 \rightarrow 0\} = -\Gamma_1$$

$$\Gamma_4 = \{\sigma(0, \theta) = (0, 0, 0) : \theta : 2\pi \rightarrow 0\}$$

We therefore deduce that $\partial\Sigma = \Gamma_2$ and that it is positively oriented. We then obtain

$$\begin{aligned}\int_{\partial\Sigma} F \cdot dl &= \int_0^{2\pi} (\cos^2 \theta \sin \theta, 1, \cos \theta) \cdot (-\sin \theta, \cos \theta, 0) d\theta \\ &= -\int_0^{2\pi} \cos^2 \theta \sin^2 \theta d\theta = -\frac{\pi}{4}\end{aligned}$$

Exercise 2.

(i) Calculation of $\iint_{\Sigma} \text{curl } F \cdot ds$. We switch to spherical coordinates and write, for $A = (0, \frac{\pi}{2}) \times (\frac{\pi}{6}, \frac{\pi}{3})$,

$$\Sigma = \{\sigma(\theta, \varphi) = (2 \cos \theta \sin \varphi, 2 \sin \theta \sin \varphi, 2 \cos \varphi) \text{ with } (\theta, \varphi) \in \bar{A}\}$$

We find

$$\sigma_\theta \wedge \sigma_\varphi = \begin{vmatrix} e_1 & e_2 & e_3 \\ -2 \sin \theta \sin \varphi & 2 \cos \theta \sin \varphi & 0 \\ 2 \cos \theta \cos \varphi & 2 \sin \theta \cos \varphi & -2 \sin \varphi \end{vmatrix} = \begin{pmatrix} -4 \cos \theta \sin^2 \varphi \\ -4 \sin \theta \sin^2 \varphi \\ -4 \cos \varphi \sin \varphi \end{pmatrix}$$

Since

$$\text{curl } F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z^2 & 0 \end{vmatrix} = \begin{pmatrix} -2z \\ 0 \\ 0 \end{pmatrix}$$

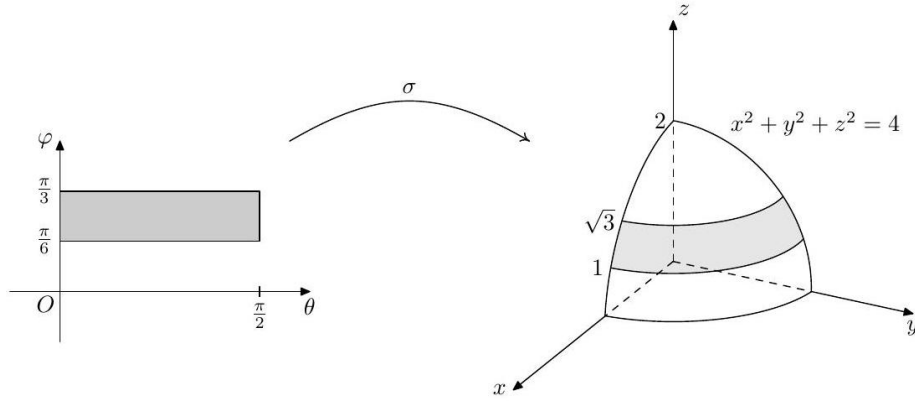
we obtain

$$\begin{aligned}\text{curl } F \cdot (\sigma_\theta \wedge \sigma_\varphi) &= (-4 \cos \varphi, 0, 0) \cdot (-4 \cos \theta \sin^2 \varphi, -4 \sin \theta \sin^2 \varphi, -4 \cos \varphi \sin \varphi) \\ &= 16 \cos \varphi \sin^2 \varphi (\cos \theta + \sin \theta)\end{aligned}$$

and therefore

$$\iint_{\Sigma} \text{curl } F \cdot ds = 16 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_0^{\frac{\pi}{2}} \cos \theta \cos \varphi \sin^2 \varphi d\theta d\varphi = 2\sqrt{3} - \frac{2}{3}$$

(ii) Calculation of $\int_{\partial\Sigma} F \cdot dl$. We have $\partial\Sigma = \sigma(\partial A) = \bigcup_{i=1}^4 \Gamma_i$, where



$$\begin{aligned}\Gamma_1 &= \left\{ \sigma \left(\theta, \frac{\pi}{6} \right) = (\cos \theta, \sin \theta, \sqrt{3}) : \theta : 0 \rightarrow \frac{\pi}{2} \right\} \\ \Gamma_2 &= \left\{ \sigma \left(\frac{\pi}{2}, \varphi \right) = (0, 2 \sin \varphi, 2 \cos \varphi) : \varphi : \frac{\pi}{6} \rightarrow \frac{\pi}{3} \right\} \\ \Gamma_3 &= \left\{ \sigma \left(\theta, \frac{\pi}{3} \right) = (\sqrt{3} \cos \theta, \sqrt{3} \sin \theta, 1) : \theta : \frac{\pi}{2} \rightarrow 0 \right\} \\ \Gamma_4 &= \left\{ \sigma(0, \varphi) = (2 \sin \varphi, 0, 2 \cos \varphi) : \varphi : \frac{\pi}{3} \rightarrow \frac{\pi}{6} \right\}\end{aligned}$$

A straightforward calculation gives

$$\begin{aligned}\int_{\Gamma_1} F \cdot dl &= \int_0^{\frac{\pi}{2}} (0, 3, 0) \cdot (-\sin \theta, \cos \theta, 0) d\theta = \int_0^{\frac{\pi}{2}} 3 \cos \theta d\theta = 3 \\ \int_{\Gamma_2} F \cdot dl &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (0, 4 \cos^2 \varphi, 0) \cdot (0, 2 \cos \varphi, -2 \sin \varphi) d\varphi \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 8 \cos^3 \varphi d\varphi = 3\sqrt{3} - \frac{11}{3} \\ \int_{\Gamma_3} F \cdot dl &= - \int_0^{\frac{\pi}{2}} (0, 1, 0) \cdot (-\sqrt{3} \sin \theta, \sqrt{3} \cos \theta, 0) d\theta = -\sqrt{3} \\ \int_{\Gamma_4} F \cdot dl &= - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (0, 4 \cos^2 \varphi, 0) \cdot (2 \cos \varphi, 0, -2 \sin \varphi) d\varphi = 0\end{aligned}$$

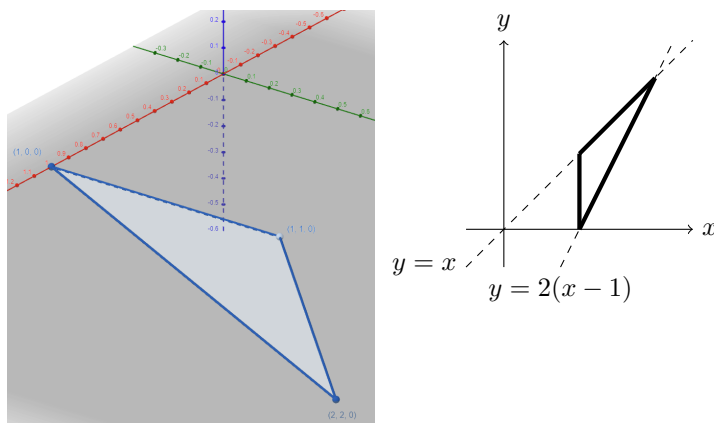
Thus, we indeed have

$$\int_{\partial \Sigma} F \cdot dl = \sum_{i=1}^4 \int_{\Gamma_i} F \cdot dl = 2\sqrt{3} - \frac{2}{3}$$

Exercise 3.(i) Computation of $\iint_{\Sigma} \text{curl } F \, ds$.

We have

$$\text{curl } F(x, y, z) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^2 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2x \end{pmatrix}$$

*Sketch of Σ :*We observe that Σ lies in the plane $z = 0$:

Notice that the domain in \mathbb{R}^2 is y -simple. It could also be split into two x -simple domains, but in that case one must take care to keep the same orientation of the normal vector for both parts of the integral. We therefore present only the y -simple variant. *Computation:*

We parameterize Σ by

$$\sigma: \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 2, 2(x-1) \leq y \leq x\} \rightarrow \mathbb{R}^3, \quad \sigma(x, y) = (x, y, 0)$$

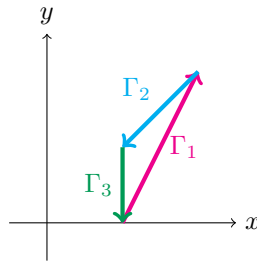
We have

$$\begin{aligned} \text{curl } F(\sigma(x, y)) &= (0, 0, 2x) \\ \sigma_x \wedge \sigma_y &= \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
\iint_{\Sigma} \operatorname{curl} F \, ds &= \int_1^2 \int_{2x-2}^x \langle (0, 0, 2x), (0, 0, 1) \rangle \, dy \, dx \\
&= \int_1^2 2x(x - (2x - 2)) \, dx \\
&= \int_1^2 (4x - 2x^2) \, dx \\
&= \left[2x^2 - \frac{2}{3}x^3 \right]_{x=1}^{x=2} \\
&= 8 - \frac{16}{3} - 2 + \frac{2}{3} \\
&= \frac{24 - 16 - 6 + 2}{3} = \frac{4}{3}
\end{aligned}$$

(ii) Computation of $\int_{\partial\Sigma} F \, dl$.

The boundary of $A = \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 2, 2(x-1) \leq y \leq x\}$ is divided into three parts:



Γ_1

$$\gamma_1: [1, 2] \rightarrow \mathbb{R}^2$$

\oplus

$$\gamma_1(t) = (t, 2t - 2)$$

$$\sigma \circ \gamma_1(t) = (t, 2t - 2, 0)$$

$$(\sigma \circ \gamma_1)'(t) = (1, 2, 0)$$

$$\begin{aligned}
\boxed{\Gamma_2} \quad & \gamma_2: [1, 2] \rightarrow \mathbb{R}^2 && \ominus \\
& \gamma_2(t) = (t, t) \\
& \sigma \circ \gamma_2(t) = (t, t, 0) \\
& (\sigma \circ \gamma_2)'(t) = (1, 1, 0)
\end{aligned}$$

$$\begin{aligned}
\boxed{\Gamma_3} \quad & \gamma_3: [0, 1] \rightarrow \mathbb{R}^2 && \ominus \\
& \gamma_3(t) = (1, t) \\
& \sigma \circ \gamma_3(t) = (1, t, 0) \\
& (\sigma \circ \gamma_3)'(t) = (0, 1, 0)
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{\sigma(\Gamma_1)} F \cdot dl &= \int_1^2 \langle (0, t^2, 0), (1, 2, 0) \rangle dt = \int_1^2 2t^2 dt = \left[\frac{2}{3} t^3 \right]_{t=1}^{t=2} \\
&= \frac{16}{3} - \frac{2}{3} = \frac{14}{3}
\end{aligned}$$

$$\begin{aligned}
\int_{\sigma(\Gamma_2)} F \cdot dl &= \int_1^2 \langle (0, t^2, 0), (1, 1, 0) \rangle dt = \int_1^2 t^2 dt = \left[\frac{1}{3} t^3 \right]_{t=1}^{t=2} \\
&= \frac{8}{3} - \frac{1}{3} = \frac{7}{3}
\end{aligned}$$

$$\int_{\sigma(\Gamma_3)} F \cdot dl = \int_0^1 \langle (0, 1, 0), (0, 1, 0) \rangle dt = \int_0^1 1 dt = 1$$

$$\begin{aligned}
\int_{\partial\Sigma} F \cdot dl &= + \int_{\sigma(\Gamma_1)} F \cdot dl - \int_{\sigma(\Gamma_2)} F \cdot dl - \int_{\sigma(\Gamma_3)} F \cdot dl \\
&= \frac{14}{3} - \frac{7}{3} - 1 = \frac{4}{3},
\end{aligned}$$

which is the expected result.

Exercise 4.(i) Computation of $\iint_{\Sigma} \operatorname{curl} F \, ds$.

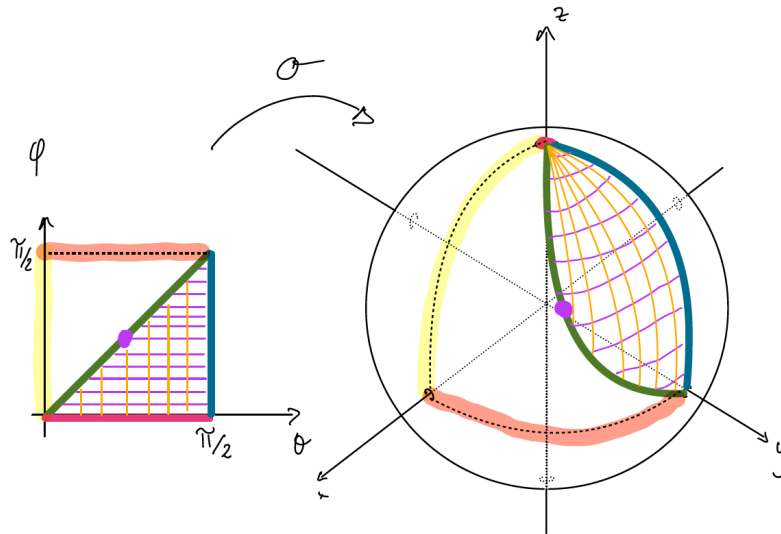
We have

$$\operatorname{curl} F(x, y, z) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & y + z^2 \end{vmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Sketch of Σ :

We switch to spherical coordinates $(x, y, z) = (r \cos \theta \sin \varphi, r \sin \theta \sin \varphi, r \cos \varphi)$ with $r \geq 0$, $\theta \in [-\pi, \pi]$, and $\varphi \in [0, \pi]$. Our conditions become

$$\begin{aligned} x^2 + y^2 + z^2 = 4 &\Leftrightarrow r = 2 \\ x, y \geq 0 &\Leftrightarrow \theta \in \left[0, \frac{\pi}{2}\right] \\ z \geq 0 &\Leftrightarrow \varphi \in \left[0, \frac{\pi}{2}\right] \\ 0 \leq \arccos \frac{z}{2} \leq \arctan \frac{y}{x} \leq \frac{\pi}{2} &\Leftrightarrow 0 \leq \varphi \leq \theta \leq \frac{\pi}{2} \end{aligned}$$

*Calculation:*We parametrize Σ with

$$\sigma: \left\{ (\theta, \varphi) \in \mathbb{R}^2 : 0 \leq \varphi \leq \theta \leq \frac{\pi}{2} \right\} \rightarrow \mathbb{R}^3$$

defined by

$$\sigma(\theta, \varphi) = (2 \cos \theta \sin \varphi, 2 \sin \theta \sin \varphi, 2 \cos \varphi)$$

We have

$$\operatorname{curl} F(\sigma(\theta, \varphi)) = (1, 0, 0)$$

$$\sigma_\theta \wedge \sigma_\varphi = \begin{vmatrix} e_1 & e_2 & e_3 \\ -2 \sin \theta \sin \varphi & 2 \cos \theta \sin \varphi & 0 \\ 2 \cos \theta \cos \varphi & 2 \sin \theta \cos \varphi & -2 \sin \varphi \end{vmatrix} = \begin{pmatrix} -4 \cos \theta \sin^2 \varphi \\ -4 \sin \theta \sin^2 \varphi \\ -4 \sin \varphi \cos \varphi \end{pmatrix}$$

Variant 1: $d\varphi d\theta$

$$\begin{aligned} \int_{\Sigma} \operatorname{curl} F ds &= \int_0^{\frac{\pi}{2}} \int_0^\theta \langle (1, 0, 0), (-4 \cos \theta \sin^2 \varphi, -4 \sin \theta \sin^2 \varphi, -4 \sin \varphi \cos \varphi) \rangle d\varphi d\theta \\ &= -4 \int_0^{\frac{\pi}{2}} \cos \theta \int_0^\theta \sin^2 \varphi d\varphi d\theta \\ &= -4 \int_0^{\frac{\pi}{2}} \cos \theta \left[\frac{1}{2} (\varphi - \sin \varphi \cos \varphi) \right]_{\varphi=0}^{\varphi=\theta} d\theta \\ &= -4 \int_0^{\frac{\pi}{2}} \cos \theta \frac{1}{2} (\theta - \sin \theta \cos \theta) d\theta \\ &= -2 \int_0^{\frac{\pi}{2}} \theta \cos \theta - \sin \theta \cos^2 \theta d\theta \\ &= -2 \int_0^{\frac{\pi}{2}} \theta \cos \theta d\theta - \left[\frac{2}{3} \cos^3 \theta \right]_0^{\frac{\pi}{2}} \\ &= -2 \int_0^{\frac{\pi}{2}} \theta \cos \theta d\theta + \frac{2}{3} \\ &\stackrel{\text{IBP}}{=} -2 \left([\theta \sin \theta]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin \theta d\theta \right) + \frac{2}{3} \\ &= -\pi - 2 [\cos \theta]_0^{\frac{\pi}{2}} + \frac{2}{3} \\ &= -\pi + 2 + \frac{2}{3} \\ &= \frac{8}{3} - \pi \end{aligned} \quad \left| \begin{array}{l} u = \theta \quad v = \sin \theta \\ u' = 1 \quad v' = \cos \theta \end{array} \right.$$

Variant 2: $d\theta d\varphi$

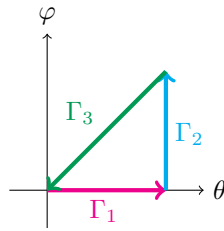
$$\begin{aligned}
 \int_{\Sigma} \operatorname{curl} F ds &= \int_0^{\frac{\pi}{2}} \int_{\varphi}^{\frac{\pi}{2}} \langle (1, 0, 0), (-4 \cos \theta \sin^2 \varphi, -4 \sin \theta \sin^2 \varphi, -4 \sin \varphi \cos \varphi) \rangle d\theta d\varphi \\
 &= -4 \int_0^{\frac{\pi}{2}} \sin^2 \varphi \int_{\varphi}^{\frac{\pi}{2}} \cos \theta d\theta d\varphi \\
 &= -4 \int_0^{\frac{\pi}{2}} \sin^2 \varphi [\sin \theta]_{\theta=\varphi}^{\theta=\frac{\pi}{2}} d\varphi \\
 &= -4 \int_0^{\frac{\pi}{2}} \sin^2 \varphi - \sin^3 \varphi d\varphi \\
 &= -4 \int_0^{\frac{\pi}{2}} \sin^2 \varphi - \sin \varphi (1 - \cos^2 \varphi) d\varphi \\
 &= 4 \int_0^{\frac{\pi}{2}} -\sin^2 \varphi + \sin \varphi - \cos^2 \varphi \sin \varphi d\varphi \\
 &= 4 \left[-\frac{1}{2} \varphi + \frac{1}{2} \cos \varphi \sin \varphi - \cos \varphi + \frac{1}{3} \cos^3 \varphi \right]_0^{\frac{\pi}{2}} \\
 &= -\pi + 4 - \frac{4}{3} = \frac{8}{3} - \pi
 \end{aligned}$$

One could also find the antiderivatives of trigonometric functions using Euler's formulas.

—

(ii) Computation of $\int_{\partial\Sigma} F dl$.

The boundary of $A = \{(\theta, \varphi) \in \mathbb{R}^2 : 0 \leq \varphi \leq \theta \leq \frac{\pi}{2}\}$ is divided into three parts:



Γ_1

$$\gamma_1 : [0, \frac{\pi}{2}] \rightarrow \mathbb{R}^2$$

$$\gamma_1(t) = (t, 0)$$

$$\sigma \circ \gamma_1(t) = (0, 0, 2)$$

is a single point, so it contributes nothing.

\oplus

$$\begin{aligned}
\boxed{\Gamma_2} \quad \gamma_2: [0, \frac{\pi}{2}] &\rightarrow \mathbb{R}^2 && \oplus \\
\gamma_2(t) &= (\frac{\pi}{2}, t) \\
\sigma \circ \gamma_2(t) &= (0, 2 \sin t, 2 \cos t) \\
(\sigma \circ \gamma_2)'(t) &= (0, 2 \cos t, -2 \sin t)
\end{aligned}$$

$$\begin{aligned}
\boxed{\Gamma_3} \quad \gamma_3: [0, \frac{\pi}{2}] &&& \ominus \\
\gamma_3(t) &= (t, t) \\
\sigma \circ \gamma_3(t) &= (2 \cos t \sin t, 2 \sin^2 t, 2 \cos t) \\
(\sigma \circ \gamma_3)'(t) &= (-2 \sin^2 t + 2 \cos^2 t, 4 \sin t \cos t, -2 \sin t)
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{\sigma(\Gamma_2)} F \cdot dl &= \int_0^{\frac{\pi}{2}} \langle (0, 0, 2 \sin t + 4 \cos^2 t), (0, 2 \cos t, -2 \sin t) \rangle dt \\
&= \int_0^{\frac{\pi}{2}} -4 \sin^2 t - 8 \sin t \cos^2 t dt \\
&= \left[-2(t - \sin t \cos t) + \frac{8}{3} \cos^3 t \right]_0^{\frac{\pi}{2}} \\
&= -\pi - \frac{8}{3}
\end{aligned}$$

$$\begin{aligned}
\int_{\sigma(\Gamma_3)} F \cdot dl &= \int_0^{\frac{\pi}{2}} \langle (0, 0, 2 \sin^2 t + 4 \cos^2 t), (-2 \sin^2 t + 2 \cos^2 t, 4 \sin t \cos t, -2 \sin t) \rangle dt \\
&= -4 \int_0^{\frac{\pi}{2}} \sin^3 t + 2 \sin t \cos^2 t dt \\
&= -4 \int_0^{\frac{\pi}{2}} \sin t - \sin t \cos^2 t + 2 \sin t \cos^2 t dt \\
&= -4 \int_0^{\frac{\pi}{2}} \sin t + \sin t \cos^2 t dt \\
&= -4 \left[-\cos t - \frac{1}{3} \cos^3 t \right]_0^{\frac{\pi}{2}} \\
&= -4 - \frac{4}{3} = -\frac{16}{3}
\end{aligned}$$

$$\int_{\Sigma} F \cdot dl = + \int_{\sigma(\Gamma_2)} F \cdot dl - \int_{\sigma(\Gamma_3)} F \cdot dl = \frac{8}{3} - \pi$$

which indeed matches the expected result.