

**Exercise 1.**

(i) Calculation of  $\iiint_{\Omega} \operatorname{div} F dx dy dz$ . We have  $\operatorname{div} F = 2(x + y + z)$ . We set

$$x = ar \cos \theta, \quad y = ar \sin \theta, \quad z = bt, \quad \text{with } 0 < \theta < 2\pi, \quad 0 < r < t < 1$$

and we therefore find

$$\text{Jacobian} = \begin{vmatrix} a \cos \theta & -ar \sin \theta & 0 \\ a \sin \theta & ar \cos \theta & 0 \\ 0 & 0 & b \end{vmatrix} = a^2 br$$

This allows us to compute

$$\begin{aligned} \iiint_{\Omega} \operatorname{div} F dx dy dz &= 2 \int_0^{2\pi} d\theta \int_0^1 dt \int_0^t a^2 br (ar \cos \theta + ar \sin \theta + bt) dr \\ &= 4\pi a^2 b^2 \int_0^1 dt \int_0^t r t dr = \frac{\pi a^2 b^2}{2} \end{aligned}$$

(ii) Calculation of  $\iint_{\partial\Omega} (F \cdot \nu) ds$ . We have  $\partial\Omega = \Sigma_1 \cup \Sigma_2$  where

$$\begin{aligned} \Sigma_1 &= \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq a^2, z = b\} \\ &= \{\alpha(r, \theta) = (r \cos \theta, r \sin \theta, b) : 0 \leq \theta \leq 2\pi, 0 \leq r \leq a\} \\ \Sigma_2 &= \{(x, y, z) \in \mathbb{R}^3 : b^2(x^2 + y^2) = a^2 z^2 \text{ and } 0 \leq z \leq b\} \\ &= \{\beta(\theta, t) = (at \cos \theta, at \sin \theta, bt) : 0 \leq \theta \leq 2\pi, 0 \leq t \leq 1\} \end{aligned}$$

We find

$$\alpha_r \wedge \alpha_\theta = \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$

$$\beta_\theta \wedge \beta_t = \begin{vmatrix} e_1 & e_2 & e_3 \\ -at \sin \theta & at \cos \theta & 0 \\ a \cos \theta & a \sin \theta & b \end{vmatrix} = \begin{pmatrix} abt \cos \theta \\ abt \sin \theta \\ -a^2 t \end{pmatrix}$$

which are both outward normals. We therefore obtain

$$\begin{aligned}
\iint_{\Sigma_1} (F \cdot \nu) ds &= \int_0^{2\pi} \int_0^a (r^2 \cos^2 \theta, r^2 \sin^2 \theta, b^2) \cdot (0, 0, r) dr d\theta = \pi a^2 b^2 \\
\iint_{\Sigma_2} (F \cdot \nu) ds &= \int_0^{2\pi} \int_0^1 (a^2 t^2 \cos^2 \theta, a^2 t^2 \sin^2 \theta, b^2 t^2) \cdot (abt \cos \theta, abt \sin \theta, -a^2 t) dt d\theta \\
&= -2\pi a^2 b^2 \int_0^1 t^3 dt = -\frac{\pi}{2} a^2 b^2
\end{aligned}$$

and finally

$$\iint_{\partial\Omega} (F \cdot \nu) ds = \frac{\pi}{2} a^2 b^2$$

**Exercise 2.**

(i) Calculation of  $\iiint_{\Omega} \operatorname{div} F dx dy dz$ . We immediately have

$$\operatorname{div} F = x + y + z.$$

We have

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : 0 < z < 1 - x - y, 0 < y < 1 - x, 0 < x < 1\}$$

and therefore

$$\begin{aligned}
\iiint_{\Omega} \operatorname{div} F dx dy dz &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x + y + z) dz dy dx \\
&= \frac{1}{2} \int_0^1 \int_0^{1-x} (1 - (x + y)^2) dy dx = \frac{1}{8}
\end{aligned}$$

(ii) Calculation of  $\iint_{\partial\Omega} (F \cdot \nu) ds$ . We have  $\partial\Omega = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$ , where

$$\begin{aligned}
\Sigma_1 &= \{\alpha(x, z) = (x, 0, z) : 0 \leq z \leq 1 - x, 0 \leq x \leq 1\} \\
\Sigma_2 &= \{\beta(x, y) = (x, y, 0) : 0 \leq y \leq 1 - x, 0 \leq x \leq 1\} \\
\Sigma_3 &= \{\gamma(y, z) = (0, y, z) : 0 \leq z \leq 1 - y, 0 \leq y \leq 1\} \\
\Sigma_4 &= \{\delta(x, y) = (x, y, 1 - x - y) : 0 \leq y \leq 1 - x, 0 \leq x \leq 1\}
\end{aligned}$$

The corresponding normals are

$$\begin{aligned}\alpha_x \wedge \alpha_z &= \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \\ \beta_x \wedge \beta_y &= \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \gamma_y \wedge \gamma_z &= \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \delta_x \wedge \delta_y &= \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\end{aligned}$$

which are outward normals for the first and last, and inward normals for the other two. Note then that

$$\iint_{\Sigma_1} (F \cdot \nu) ds = \iint_{\Sigma_2} (F \cdot \nu) ds = \iint_{\Sigma_3} (F \cdot \nu) ds = 0$$

and therefore

$$\begin{aligned}\iint_{\partial\Omega} (F \cdot \nu) ds &= \iint_{\Sigma_4} (F \cdot \nu) ds \\ &= \int_0^1 \int_0^{1-x} ((xy, y(1-x-y), x(1-x-y)) \cdot (1, 1, 1)) dy dx\end{aligned}$$

which finally gives us

$$\iint_{\partial\Omega} (F \cdot \nu) ds = \frac{1}{8}$$

**Exercise 3.**

(i) Calculation of  $\iiint_{\Omega} \operatorname{div} F dx dy dz$ . We immediately find that  $\operatorname{div} F = 3$ . We switch to cylindrical coordinates and obtain

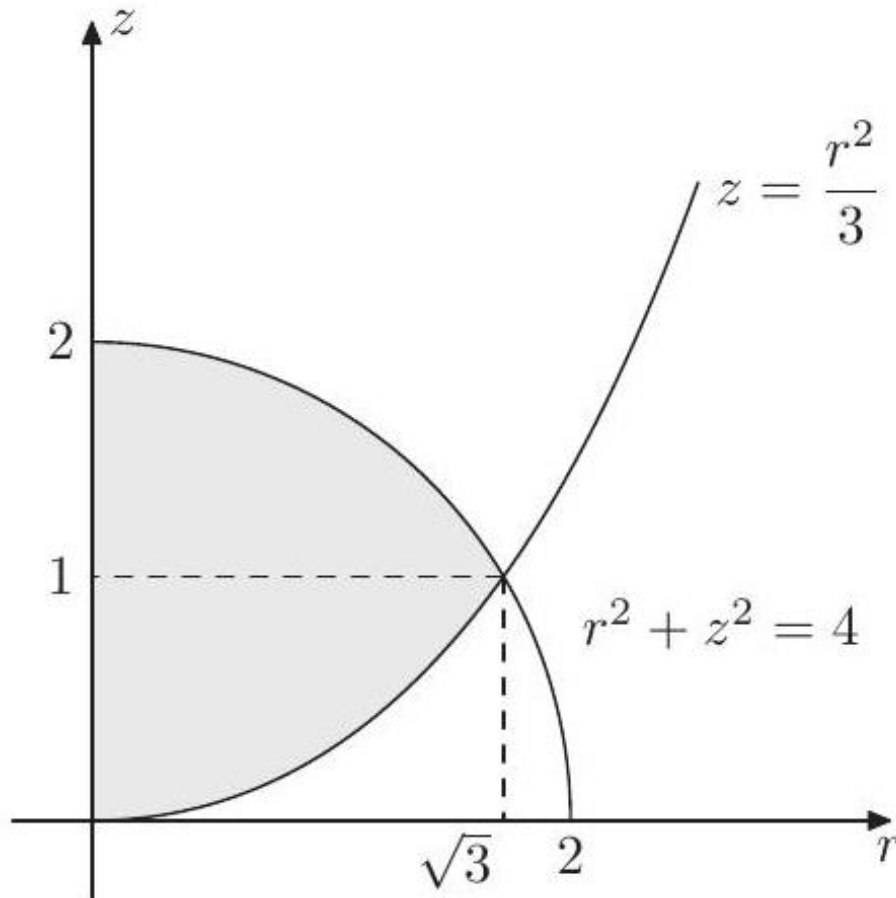
$$\Omega = \left\{ (r \cos \theta, r \sin \theta, z) : \theta \in (0, 2\pi), r \in (0, \sqrt{3}), \frac{r^2}{3} < z < \sqrt{4-r^2} \right\}$$

and therefore

$$\begin{aligned} \iiint_{\Omega} \operatorname{div} F dx dy dz &= 3 \int_0^{\sqrt{3}} \int_{\frac{r^2}{3}}^{\sqrt{4-r^2}} \int_0^{2\pi} r d\theta dz dr \\ &= 6\pi \int_0^{\sqrt{3}} \left( \sqrt{4-r^2} - \frac{r^2}{3} \right) r dr = \frac{19\pi}{2} \end{aligned}$$

(ii) Calculation of  $\iint_{\partial\Omega} (F \cdot \nu) ds$ . Note that  $\partial\Omega = \Sigma_1 \cup \Sigma_2$  where

$$\Sigma_1 = \left\{ \alpha(\theta, \varphi) = (2 \cos \theta \sin \varphi, 2 \sin \theta \sin \varphi, 2 \cos \varphi) : \theta \in [0, 2\pi], \varphi \in \left[0, \frac{\pi}{3}\right] \right\}$$



(dessin dans le plan  $r, z$ )

$$\Sigma_2 = \left\{ \beta(r, \theta) = \left( r \cos \theta, r \sin \theta, \frac{r^2}{3} \right) : \theta \in [0, 2\pi], r \in [0, \sqrt{3}] \right\}$$

The calculation of the normals gives

$$\alpha_\theta \wedge \alpha_\varphi = \begin{vmatrix} e_1 & e_2 & e_3 \\ -2 \sin \theta \sin \varphi & 2 \cos \theta \sin \varphi & 0 \\ 2 \cos \theta \cos \varphi & 2 \sin \theta \cos \varphi & -2 \sin \varphi \end{vmatrix} = \begin{pmatrix} -4 \cos \theta \sin^2 \varphi \\ -4 \sin \theta \sin^2 \varphi \\ -4 \cos \varphi \sin \varphi \end{pmatrix}$$

$$\beta_r \wedge \beta_\theta = \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos \theta & \sin \theta & \frac{2}{3}r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = \begin{pmatrix} -\frac{2}{3}r^2 \cos \theta \\ -\frac{2}{3}r^2 \sin \theta \\ r \end{pmatrix}$$

which are both inward normals. We can therefore compute the surface integrals

$$\begin{aligned} \iint_{\Sigma_1} (F \cdot \nu) ds &= \int_0^{2\pi} \int_0^{\pi/3} [(2 \cos \theta \sin \varphi, 2 \sin \theta \sin \varphi, 2 \cos \varphi) \\ &\quad (4 \cos \theta \sin^2 \varphi, 4 \sin \theta \sin^2 \varphi, 4 \cos \varphi \sin \varphi)] d\theta d\varphi \\ &= 16\pi \int_0^{\pi/3} \sin \varphi d\varphi = 16\pi [-\cos \varphi]_0^{\pi/3} = 8\pi \\ \iint_{\Sigma_2} (F \cdot \nu) ds &= \int_0^{2\pi} \int_0^{\sqrt{3}} \left( r \cos \theta, r \sin \theta, \frac{r^2}{3} \right) \cdot \left( \frac{2r^2 \cos \theta}{3}, \frac{2r^2 \sin \theta}{3}, -r \right) dr d\theta \\ &= 2\pi \int_0^{\sqrt{3}} \left( \frac{2r^3}{3} - \frac{r^3}{3} \right) dr = 2\pi \left[ \frac{r^4}{12} \right]_0^{\sqrt{3}} = \frac{3\pi}{2} \end{aligned}$$

We therefore find

$$\iint_{\partial\Omega} (F \cdot \nu) ds = 8\pi + \frac{3\pi}{2} = \frac{19\pi}{2}$$

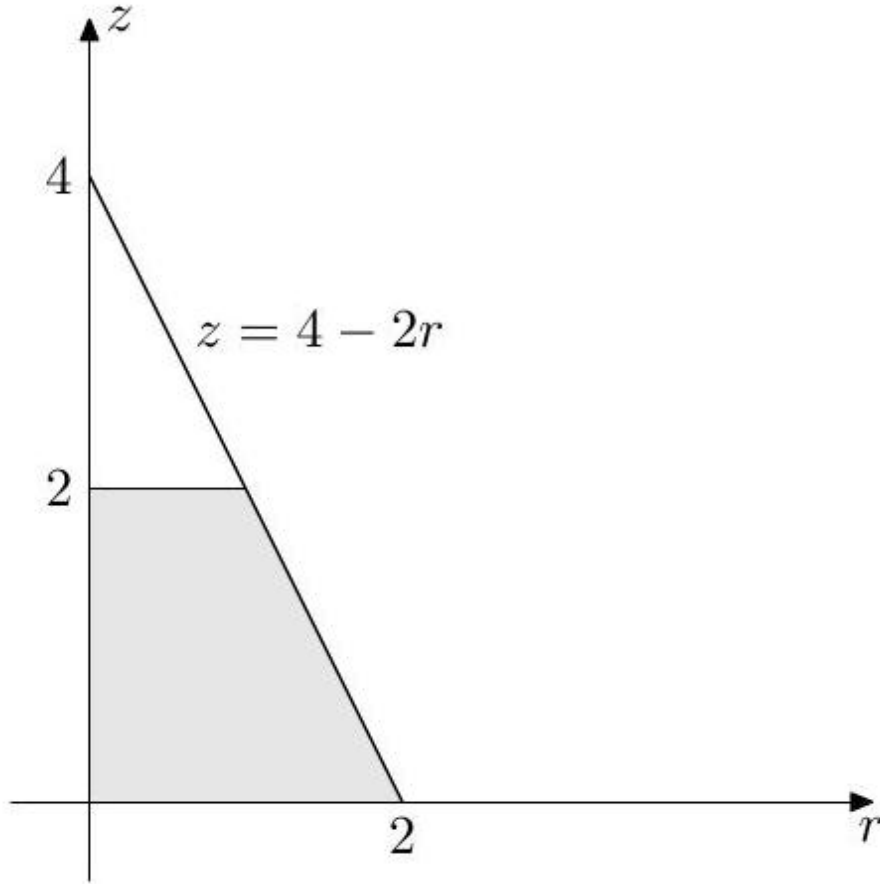
**Exercise 4.**

(i) Calculation of  $\iiint_{\Omega} \operatorname{div} F dx dy dz$ . We find that  $\operatorname{div} F = 2z$ . Switching to cylindrical coordinates, we deduce that

$$\Omega = \left\{ (r \cos \theta, r \sin \theta, z) : 0 < z < 2, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, 0 < 2r < 4 - z \right\}$$

and therefore

$$\begin{aligned} \iiint_{\Omega} \operatorname{div} F dx dy dz &= \int_0^2 \int_0^{2-z/2} \int_{-\pi/2}^{\pi/2} 2zr d\theta dz dr \\ &= \pi \int_0^2 z \left(2 - \frac{z}{2}\right)^2 dz = \frac{11\pi}{3} \end{aligned}$$



(drawing in the  $r, z$  plane with rotation angle  $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ )

(ii) Calculation of  $\iint_{\partial\Omega} (F \cdot \nu) ds$ . We see that  $\partial\Omega = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$  with

$$\begin{aligned} \Sigma_1 &= \{\alpha(y, z) = (0, y, z) : z \in (0, 2) \text{ and } 2|y| \leq 4 - z\} \\ \Sigma_2 &= \left\{ \beta(r, \theta) = (r \cos \theta, r \sin \theta, 0) : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, r \leq 2 \right\} \\ \Sigma_3 &= \left\{ \gamma(r, \theta) = (r \cos \theta, r \sin \theta, 2) : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, r \leq 1 \right\} \\ \Sigma_4 &= \left\{ \delta(r, \theta) = (r \cos \theta, r \sin \theta, 4 - 2r) : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq 2 \right\}. \end{aligned}$$

The normals are then

$$\begin{aligned}\alpha_y \wedge \alpha_z &= (1, 0, 0), & \beta_r \wedge \beta_\theta &= (0, 0, r) \\ \gamma_r \wedge \gamma_\theta &= (0, 0, r), & \delta_r \wedge \delta_\theta &= (2r \cos \theta, 2r \sin \theta, r)\end{aligned}$$

and these are inward normals for the first two and outward normals for the last two. Consequently, we find

$$\begin{aligned}\iint_{\Sigma_1} (F \cdot \nu) ds &= \int_0^2 \int_{-2+z/2}^{2-z/2} (2, 0, z^2) \cdot (-1, 0, 0) dy dz = -12 \\ \iint_{\Sigma_2} (F \cdot \nu) ds &= \int_{-\pi/2}^{\pi/2} \int_0^2 (2, 0, r^3 \cos \theta \sin^2 \theta) \cdot (0, 0, -r) dr d\theta = -\frac{64}{15} \\ \iint_{\Sigma_3} (F \cdot \nu) ds &= \int_{-\pi/2}^{\pi/2} \int_0^1 (2, 0, r^3 \cos \theta \sin^2 \theta + 4) \cdot (0, 0, r) dr d\theta \\ &= \int_{-\pi/2}^{\pi/2} \int_0^1 (r^4 \cos \theta \sin^2 \theta + 4r) dr d\theta = \frac{2}{15} + 2\pi \\ \iint_{\Sigma_4} (F \cdot \nu) ds &= \int_{-\pi/2}^{\pi/2} \int_1^2 (2, 0, r^3 \cos \theta \sin^2 \theta + (4 - 2r)^2) \cdot (2r \cos \theta, 2r \sin \theta, r) dr d\theta \\ &= \int_{-\pi/2}^{\pi/2} \int_1^2 (4r \cos \theta + r^4 \cos \theta \sin^2 \theta + r(4 - 2r)^2) dr d\theta \\ &= 12 + \frac{62}{15} + \frac{5\pi}{3}.\end{aligned}$$

We thus obtain the desired result

$$\iint_{\partial\Omega} (F \cdot \nu) ds = \sum_{i=1}^4 \iint_{\Sigma_i} (F \cdot \nu) ds = \frac{11\pi}{3}$$

**Exercise 5.**

We first note that

$$\operatorname{div} F = 3 \quad \text{and} \quad \operatorname{div} G_i = 1, \quad i = 1, 2, 3.$$

Consequently, by applying the divergence theorem, we obtain, for  $i = 1, 2, 3$ ,

$$\begin{aligned}\text{Vol}(\Omega) &= \iiint_{\Omega} dx dy dz = \frac{1}{3} \iiint_{\Omega} \text{div } F dx dy dz = \frac{1}{3} \iint_{\partial\Omega} (F \cdot \nu) ds \\ &= \iiint_{\Omega} \text{div } G_i dx dy dz = \iint_{\partial\Omega} (G_i \cdot \nu) ds\end{aligned}$$