Exercise 1.

1. Let us note that

$$\operatorname{rot}\Phi = \frac{\partial}{\partial x}[\Phi_2] - \frac{\partial}{\partial y}[\Phi_1] = \frac{\partial}{\partial x}[F_1] - \frac{\partial}{\partial y}[-F_2] = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \operatorname{div} F.$$

Thus, by Green's theorem,

$$\iint_\Omega \operatorname{div} F(x,y)dxdy = \iint_\Omega \operatorname{rot}\Phi(x,y)dxdy = \int_{\partial\Omega} \Phi \cdot dl$$

Moreover, if $\gamma = (\gamma_1, \gamma_2)\colon [a,b] \to \mathbb{R}^2$ is a parametrization that keeps the domain on the left (i.e. $\partial\Omega$ is positively oriented), we have

$$\begin{aligned}\int_{\partial\Omega} \Phi \cdot dl &= \int_a^b \langle \Phi(\gamma(t)), \gamma'(t)\rangle\, dt = \int_a^b \langle (-F_2(\gamma(t)), F_1(\gamma(t))), (\gamma_1'(t), \gamma_2'(t))\rangle\, dt \\ &= \int_a^b F_1(\gamma(t))\gamma_2'(t) - F_2(\gamma(t))\gamma_1'(t)dt\end{aligned}$$

Furthermore, using the formula for computing a line integral of $\langle F, \nu\rangle$, we have

$$\begin{aligned}\int_{\partial\Omega} \langle F, \nu\rangle\, dl &= \int_a^b \langle (F_1(\gamma(t)), F_2(\gamma(t))), (\gamma_2'(t), -\gamma_1'(t))\rangle\, dt \\ &= \int_a^b F_1(\gamma(t))\gamma_2'(t) - F_2(\gamma(t))\gamma_1'(t)dt = \int_{\partial\Omega} \Phi \cdot dl\end{aligned}$$

Hence, the result follows.

2. By applying the divergence theorem to $F = \nabla f$, we have

$$\iint_\Omega \Delta f(x,y)dxdy = \int_\Omega \operatorname{div}(\nabla f)(x,y)dxdy = \int_{\partial\Omega} \langle \nabla f, \nu\rangle\, dl,$$

which is the desired result.

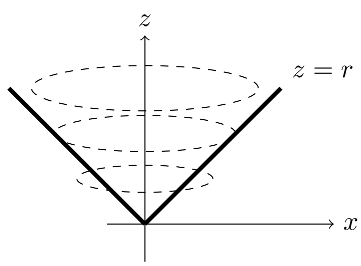
Exercise 2.
We switch to cylindrical coordinates: $(x,y,z) = (r\cos\theta, r\sin\theta, z)$, with $r \geq 0$, $\theta \in [0, 2\pi]$, and $z \in \mathbb{R}$.

Our conditions become:

$$x^2 + y^2 = z^2 \Leftrightarrow r^2 = z^2 \Leftrightarrow r = |z| \overset{z \geq 0}{=} z$$
$$0 \leq z \leq 1 \text{ remains unchanged.}$$

and therefore a parametrization of the surface is given by $\sigma\colon [0,1] \times [0, 2\pi] \to \mathbb{R}^3$ defined by

$$\sigma(r, \theta) = (r\cos\theta, r\sin\theta, r).$$



Moreover, we have

$$\begin{aligned}
\sigma_r \wedge \sigma_\theta &= \begin{vmatrix} e_1 & e_2 & e_3 \\ \sigma_r^1 & \sigma_r^2 & \sigma_r^3 \\ \sigma_\theta^1 & \sigma_\theta^2 & \sigma_\theta^3 \end{vmatrix} \\
&= \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} \\
&= e_1(-r\cos\theta) + e_2(-r\sin\theta) + e_3(r\cos^2\theta + r\sin^2\theta) \\
&= \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{pmatrix}
\end{aligned}$$

and therefore,

$$\|\sigma_r \wedge \sigma_\theta\| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2}r.$$

Finally,

$$
\begin{aligned}
f(\sigma(r,\theta)) =& r^2 \cos\theta \sin\theta + r^2 \\
\iint_\Sigma f\, ds =& \int_0^{2\pi} \int_0^1 \left(r^2 \cos\theta \sin\theta + r^2\right) \sqrt{2} r \, dr \, d\theta \\
=& \sqrt{2} \int_0^{2\pi} \int_0^1 r^3 \cos\theta \sin\theta + r^3 \, dr \, d\theta \\
=& \sqrt{2} \int_0^{2\pi} \frac{1}{4} \cos\theta \sin\theta + \frac{1}{4} \, d\theta \\
=& \frac{\sqrt{2}}{4} \left[\frac{1}{2} \sin^2\theta + \theta \right]_0^{2\pi} \\
=& \frac{\sqrt{2}\pi}{2}
\end{aligned}
$$

Exercise 3.

1. For a 3D representation of Ω, see `https://www.geogebra.org/calculator/ctqgfts5`.
2. The parametrization is $u\colon [0,a] \times [0,2\pi] \times [0,2\pi] \to \mathbb{R}^3$ defined by

$$u(r,\theta,\varphi) = ((R + r\cos\varphi)\cos\theta, (R + r\cos\varphi)\sin\theta, r\sin\varphi).$$

Thus, the Jacobian matrix of φ is

$$\nabla u(r,\theta,\varphi) = \begin{pmatrix} \cos\varphi\cos\theta & -(R + r\cos\varphi)\sin\theta & -r\sin\varphi\cos\theta \\ \cos\varphi\sin\theta & (R + r\cos\varphi)\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi & 0 & r\cos\varphi \end{pmatrix}$$

and therefore the Jacobian determinant of u is

$$
\begin{aligned}
\det \nabla u(r,\theta,\varphi) =& r(R + \cos\varphi)\cos^2\varphi\cos^2\theta + r(R + r\cos\varphi)\sin^2\varphi\sin^2\theta \\
& + r(R + r\cos\varphi)\sin^2\varphi\cos^2\theta + r(R + r\cos\varphi)\cos^2\varphi\sin^2\theta \\
=& r(R + r\cos\varphi)\left(\cos^2\varphi(\cos^2\theta + \sin^2\theta) + \sin^2\varphi(\cos^2\theta + \sin^2\theta)\right) \\
=& r(R + r\cos\varphi).
\end{aligned}
$$

Note that since $0 \le r \le a \le R$ and $\cos\varphi \in [-1,1]$, we have $R + r\cos\varphi \ge 0$, and thus

$$|\det \nabla u(r,\theta,\varphi)| = r(R + r\cos\varphi).$$

3. We have

$$\begin{aligned}\operatorname{vol}(\Omega) =& \iiint_\Omega dx\,dy\,dz = \int_0^a \int_0^{2\pi} \int_0^{2\pi} r(R + r\cos\varphi)\,d\theta\,d\varphi\,dr \\ =& 2\pi \int_0^a \left[r(R\varphi + r\sin\varphi)\right]_0^{2\pi} = 4\pi^2 R \frac{a^2}{2} = 2\pi^2 R a^2.\end{aligned}$$

4. The parametrization of the surface of the torus consists of fixing $r = a$. Thus, $\sigma\colon [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$ is defined by

$$\sigma(\theta, \varphi) = u(a, \theta, \varphi) = ((R + a\cos\varphi)\cos\theta, (R + a\cos\varphi)\sin\theta, a\sin\varphi),$$

and we have

$$\begin{aligned}\sigma_\theta(\theta, \varphi) =& (-(R + a\cos\varphi)\sin\theta, (R + a\cos\varphi)\cos\theta, 0), \\ \sigma_\varphi(\theta, \varphi) =& (-a\sin\varphi\cos\theta, -a\sin\varphi\sin\theta, a\cos\varphi), \\ \sigma_\theta \wedge \sigma_\varphi =& \begin{vmatrix} e_1 & e_2 & e_3 \\ -(R + a\cos\varphi)\sin\theta & (R + a\cos\varphi)\cos\theta & 0 \\ -a\sin\varphi\cos\theta & -a\sin\varphi\sin\theta & a\cos\varphi \end{vmatrix} \\ =& \begin{pmatrix} a(R + a\cos\varphi)\cos\varphi\cos\theta \\ a(R + a\cos\varphi)\cos\varphi\sin\theta \\ a(R + a\cos\varphi)\sin\varphi\sin^2\theta + a(R + a\cos\varphi)\sin\varphi\cos^2\theta \end{pmatrix} \\ =& a(R + a\cos\varphi)\begin{pmatrix} \cos\varphi\cos\theta \\ \cos\varphi\sin\theta \\ \sin\varphi \end{pmatrix},\end{aligned}$$

which is a normal vector to the surface.

5. We have

$$\begin{aligned}|\sigma_\theta \wedge \sigma_\varphi| =& a(R + a\cos\varphi)\sqrt{\cos^2\varphi\cos^2\theta + \cos^2\varphi\sin^2\theta + \sin^2\varphi} = a(R + a\cos\varphi), \\ \operatorname{Area}(\partial\Omega) =& \iint_{\partial\Omega} ds = \int_0^{2\pi} \int_0^{2\pi} a(R + a\cos\varphi)\,d\theta\,d\varphi \\ =& 2\pi\left[a(R\varphi + a\sin\varphi)\right]_0^{2\pi} = 4\pi^2 aR.\end{aligned}$$

6. We have

$$\begin{aligned}\iiint_\Omega z^2\,dx\,dy\,dz =& \int_0^a \int_0^{2\pi} \int_0^{2\pi} r^2\sin^2\varphi \cdot r(R + r\cos\varphi)\,d\theta\,d\varphi\,dr \\ =& 2\pi \int_0^a \int_0^{2\pi} Rr^3\sin^2\varphi + r^3\sin^2\varphi\cos\varphi\,d\varphi\,dr.\end{aligned}$$

Using that

$$\begin{aligned}\int \sin^2\varphi\,d\varphi =& \frac{1}{2}\left(\varphi - \sin\varphi\cos\varphi\right), \\ \int \sin^2\varphi\cos\varphi\,d\varphi =& \frac{1}{3}\sin^3\varphi,\end{aligned}$$

or, using Euler's formulas:

$$\begin{aligned}
\sin^2\varphi &= \left(\frac{e^{i\varphi}-e^{-i\varphi}}{2i}\right)^2 = -\frac{e^{i2\varphi}-2+e^{-i2\varphi}}{4} = \frac{1}{2}-\frac{1}{2}\cos(2\varphi),\\
\int \sin^2\varphi\, d\varphi &= \frac{\varphi}{2}-\frac{1}{4}\sin(2\varphi),\\
\sin^2\varphi\cos\varphi\, d\varphi &= \left(\frac{e^{i\varphi}-e^{-i\varphi}}{2i}\right)^2 \frac{e^{i\varphi}+e^{-i\varphi}}{2} = -\frac{e^{i2\varphi}-2+e^{-i2\varphi}}{4}\frac{e^{i\varphi}+e^{-i\varphi}}{2}\\
&= -\frac{1}{8}\left(e^{i3\varphi}-2e^{i\varphi}+e^{-i\varphi}+e^{i\varphi}-2e^{-i\varphi}+e^{-i3\varphi}\right)\\
&= -\frac{1}{8}\left(e^{i3\varphi}+e^{-i3\varphi}-e^{i\varphi}-e^{-i\varphi}\right)\\
&= \frac{1}{4}\cos\varphi-\frac{1}{4}\cos(3\varphi),\\
\int \sin^2\varphi\cos\varphi\, d\varphi &= \frac{1}{4}\sin\varphi-\frac{1}{12}\sin(3\varphi).
\end{aligned}$$

Thus, returning to our computation,

$$\iiint_\Omega z^2\,dx\,dy\,dz = 2\pi\int_0^a \left[Rr^3\frac{1}{2}\left(\varphi-\sin\varphi\cos\varphi\right)+r^3\frac{1}{3}\sin^3\varphi\right]_0^{2\pi} = 2\pi^2 R\int_0^a r^3 dr = \frac{\pi^2 Ra^4}{2}.$$

Exercise 4.

1. D_1 is the triangle with vertices $(0,0)$, $(0,1)$, and $(1,0)$. We can parameterize it as $D_1 = \left\{(x,y)\in\mathbb{R}^2 : x\in[0,1],\ y\in[0,1-x]\right\}$.

$$\begin{aligned}
\iint_{D_1}\sqrt{1-x-y}\,\mathrm{d}x\mathrm{d}y &= \int_0^1 \mathrm{d}x\int_0^{1-x}\sqrt{1-x-y}\,\mathrm{d}y\\
&= \int_0^1 \left(-\frac{2}{3}(1-x-y)^{3/2}\Big|_{y=0}^{y=1-x}\right)\mathrm{d}x\\
&= \frac{2}{3}\int_0^1 (1-x)^{3/2}\,\mathrm{d}x = \frac{2}{3}\cdot\left(-\frac{2}{5}\,(1-x)^{5/2}\Big|_{x=0}^{x=1}\right)\\
&= \frac{2}{3}\cdot\frac{2}{5} = \frac{4}{15}.
\end{aligned}$$

2. To parameterize D_2, we use polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, with $\theta\in[0,2\pi]$ and $r\in[0,2(1+\cos\theta)]$. Indeed, observe that

$$0\le x^2+y^2\le 2\left(x+\sqrt{x^2+y^2}\right)\Leftrightarrow 0\le r\le 2(1+\cos\theta).$$

Recall that for polar coordinates, the Jacobian is r.

$$\iint_{D_2} \frac{\mathrm{d}x\mathrm{d}y}{(x^2+y^2)^{\frac{3}{4}}} = \int_0^{2\pi} \mathrm{d}\theta \int_0^{2(1+\cos\theta)} \frac{1}{r^{3/2}}\, r\,\mathrm{d}r = \int_0^{2\pi} 2\sqrt{2(1+\cos\theta)}\,\mathrm{d}\theta$$

using the trigonometric identity $1+\cos\theta = 2\cos^2(\theta/2)$, we find

$$\begin{aligned} &= 4\int_0^{2\pi} |\cos(\theta/2)|\ \mathrm{d}\theta = 4\int_0^{\pi} \cos(\theta/2)\,\mathrm{d}\theta - 4\int_{\pi}^{2\pi} \cos(\theta/2)\,\mathrm{d}\theta \\ &= 8\ \sin(\theta/2)\big|_{\theta=0}^{\theta=\pi} - 8\ \sin(\theta/2)\big|_{\theta=\pi}^{\theta=2\pi} = 16. \end{aligned}$$

3. Finally, $D_3 = \left\{(x,y,z)\in\mathbb{R}^3 :\ x\in[0,1],\ y\in[0,1-x],\ z\in\left[0,1-y^2\right]\right\}$.

$$\begin{aligned} \iiint_{D_3} z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{1-y^2} z\,\mathrm{d}z = \frac{1}{2}\int_0^1 \mathrm{d}x \int_0^{1-x} (1-y^2)^2\,\mathrm{d}y \\ &= \frac{1}{2}\int_0^1 \left(y - \frac{2}{3}y^3 + \frac{1}{5}y^5 \Big|_{y=0}^{y=1-x} \right) \mathrm{d}x \\ &= \frac{1}{2}\int_0^1 \left((1-x) - \frac{2}{3}(1-x)^3 + \frac{1}{5}(1-x)^5 \right) \mathrm{d}x \\ &= \frac{1}{2}\left(-\frac{1}{2}(1-x)^2 + \frac{1}{6}(1-x)^4 - \frac{1}{30}(1-x)^6 \right)\Big|_{x=0}^{x=1} = \frac{11}{60}. \end{aligned}$$

Exercise 5.
For the two domains D_1 and D_2, we use cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$.

1. The paraboloid, obtained by rotating around the Oz axis of the curve $z = r^2$, intersects the sphere, obtained by rotating around the Oz axis of the curve $r^2+z^2 = 1$, when $z^2+z-1 = 0$ and $z > 0$, hence at $z^* = \frac{\sqrt{5}-1}{2}$. The domain $D_1 = P \cup B$ is therefore composed of a piece of paraboloid and a piece of sphere

$$P = \left\{(r\cos\theta, r\sin\theta, z)\in\mathbb{R}^3 : z\in[0,z^*],\ \theta\in[0,2\pi],\ r\in\left[0,\sqrt{z}\right]\right\},$$
$$B = \left\{(r\cos\theta, r\sin\theta, z)\in\mathbb{R}^3 : z\in[z^*,1],\ \theta\in[0,2\pi],\ r\in\left[0,\sqrt{1-z^2}\right]\right\}.$$

Its volume is given by

$$\text{Volume}(D_1) = \text{Volume}(P) + \text{Volume}(B) = \iiint\limits_P \mathrm{d}x\mathrm{d}y\mathrm{d}z + \iiint\limits_B \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \int_0^{\frac{\sqrt{5}-1}{2}} \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{z}} r\,\mathrm{d}r + \int_{\frac{\sqrt{5}-1}{2}}^{1} \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{1-z^2}} r\,\mathrm{d}r$$

$$= 2\pi \int_0^{\frac{\sqrt{5}-1}{2}} \frac{z}{2}\,\mathrm{d}z + 2\pi \int_{\frac{\sqrt{5}-1}{2}}^{1} \frac{1-z^2}{2}\,\mathrm{d}z = \frac{5\pi}{12}\left(3-\sqrt{5}\right).$$

2. For the domain D_2, the condition $x^2 + y^2 \leq 1$ translates to $0 \leq r \leq 1$. The two conditions $x \geq 0$ and $y \geq 0$ translate to $0 \leq \theta \leq \pi/2$. The last conditions $z \geq 0$ and $x+y+z \leq \sqrt{2}$ become $0 \leq z \leq \sqrt{2} - r\cos\theta - r\sin\theta$. Note, however, that for all $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi/2$, the quantity $\sqrt{2} - r\cos\theta - r\sin\theta \geq 0$, with equality if and only if $r = 1$ and $\theta = \pi/4$. Thus,

$$D_2 = \left\{(r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3 : r \in [0,1],\ \theta \in [0,\pi/2],\ z \in \left[0, \sqrt{2} - r\cos\theta - r\sin\theta\right]\right\}.$$

Its volume is given by

$$\text{Volume}(D_2) = \iiint\limits_{D_2} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 r\,\mathrm{d}r \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\sqrt{2}-r\cos\theta-r\sin\theta} \mathrm{d}z$$

$$= \int_0^1 r\,\mathrm{d}r \int_0^{\pi/2} \left(\sqrt{2} - r\cos\theta - r\sin\theta\right) \mathrm{d}\theta$$

$$= \int_0^1 r\left(\frac{\sqrt{2}\pi}{2} - r - r\right) \mathrm{d}r = \frac{\sqrt{2}\pi}{4} - \frac{2}{3}.$$