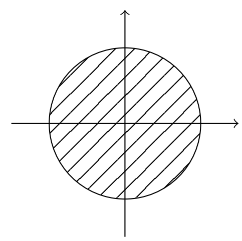
EPFL - Fall 2025	Dr. Pablo Antolin
Analysis III – MATH-202(c) – EL GM MX	**Exercises**
Correction 5	October 7 2025

Exercise 1.

1. *Curl computation* : We have

$$\operatorname{rot} F(x,y) = \frac{\partial}{\partial x}\left[y^2\right] - \frac{\partial}{\partial y}\left[xy\right] = -x.$$

Domain parametrization :



We switch to polar coordinates: $(x,y) = (r\cos\theta, r\sin\theta)$, $r \geq 0$, $\theta \in [0, 2\pi]$.

Our condition becomes $x^2 + y^2 < 1 \quad \Leftrightarrow \quad r < 1$.

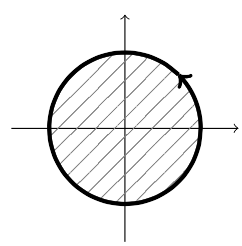
Computation of $\iint_A \operatorname{rot} F dxdy$:

The Jacobian of polar coordinates is r, and thus,

$$\begin{aligned}\iint \operatorname{rot} F dxdy &= \int_0^1 \int_0^{2\pi} \operatorname{rot} F(r\cos\theta, r\sin\theta) r d\theta dr = \int_0^1 \int_0^{2\pi} -r^2 \cos\theta d\theta dr \\ &= -\int_0^1 r^2 dr \int_0^{2\pi} \cos\theta d\theta = -\left[\frac{1}{3}r^3\right]\left[-\sin\theta\right]_0^{2\pi} = 0.\end{aligned}$$

Boundary parametrization :

The boundary is the circle centered at $(0,0)$ with radius 1. We parametrize with $\gamma\colon [0,2\pi] \to \mathbb{R}^2$ defined by $\gamma(t) = (\cos t, \sin t)$.



This parametrization leaves the domain to the left and we have

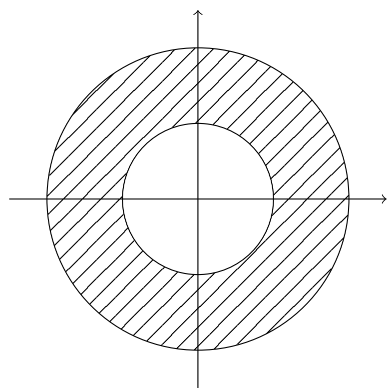
$$\begin{aligned}
F(\gamma(t)) =& (\cos t \sin t, \sin^2 t) \\
\gamma'(t) =& (-\sin t, \cos t)
\end{aligned}$$

$$\begin{aligned}
\int_{\partial A} F \cdot dl =& \int_0^{2\pi} \langle F(\gamma(t), \gamma'(t)\rangle \, dt = \int_0^{2\pi} \left\langle (\cos t \sin t, \sin^2 t), (-\sin t, \cos t) \right\rangle dt \\
=& \int_0^{2\pi} -\cos t \sin^2 t + \cos t \sin^2 t dt = \int_0^{2\pi} 0 dt = 0
\end{aligned}$$

2. *Curl computation* : We have

$$\operatorname{rot} F(x,y) = \frac{\partial}{\partial x}\left[y^2\right] - \frac{\partial}{\partial y}\left[x+y\right] = -1$$

Domain parametrization :



The domain is the annulus centered at $(0,0)$ between radii $r = 1$ and $r = 2$. We switch to polar coordinates, and our condition becomes $1 < x^2 + y^2 < 4 \quad \Leftrightarrow \quad 1 < r < 2$.

Computation of $\iint_A \operatorname{rot} F(x,y) dx dy$:

The Jacobian of polar coordinates is r and thus

$$\begin{aligned}
\iint_A \operatorname{rot} F(x,y) dx dy =& \int_1^2 \int_0^{2\pi} \operatorname{rot} F(r\cos\theta, r\sin\theta) r d\theta dr = \int_1^2 -r d\theta dr \\
=& -2\pi \left[\frac{1}{2}r^2\right]_1^2 = -2\pi\left(2 - \frac{1}{2}\right) = -3\pi.
\end{aligned}$$

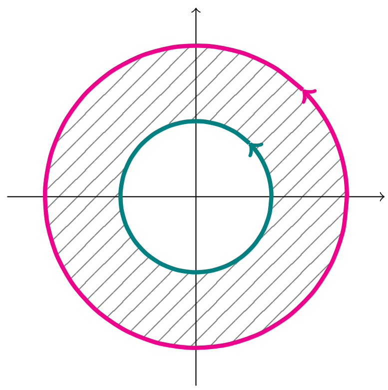
Alternatively, since $\operatorname{rot} F$ is constant, we have

$$\begin{aligned}
\iint_A \operatorname{rot} F(x,y) dx dy =& \operatorname{rot} F \cdot \operatorname{area}(A) = -\left(\operatorname{area} B((0,0),2) - \operatorname{area} B((0,0),1)\right) \\
=& -\left(\pi \cdot 4 - \pi\right) = -3\pi.
\end{aligned}$$

Parametrization of ∂A :

The boundary has two parts: $\Gamma_1 = \left\{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2=1\right\}$, the circle centered at $(0,0)$ with radius 1, and $\Gamma_2 = \left\{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2=4\right\}$, the circle centered at $(0,0)$ with radius 2, parametrized respectively by

$$\begin{aligned} \gamma_1(t) =& (\cos t, \sin t) \qquad t \in [0, 2\pi] \\ \gamma_2(t) =& (2\cos t, 2\sin t) \qquad t \in [0, 2\pi]. \end{aligned}$$



We see that γ_1 leaves the domain to the right and is thus negatively oriented, while γ_2 leaves the domain to the left and is thus positively oriented. We therefore have

$$\begin{aligned} F(\gamma_1(t)) =& (\cos t + \sin t, \sin^2 t) \qquad \ominus \\ \gamma_1'(t) =& (-\sin t, \cos t) \\ \int_{\Gamma_1} F \cdot dl =& \int_0^{2\pi} \langle F(\gamma_1(t)), \gamma_1'(t)\rangle \, dt = \int_0^{2\pi} \left\langle (\cos t + \sin t, \sin^2 t), (-\sin t, \cos t)\right\rangle dt \\ =& \int_0^{2\pi} -\cos t \sin t - \sin^2 t + \sin^2 t \cos t dt \\ =& \left[\frac{1}{2}\cos^2 t - \frac{1}{2}(t - \sin t \cos t) + \frac{1}{3}\sin^3 t\right]_0^{2\pi} = -\pi \end{aligned}$$

Alternatively, we can compute the primitives of the trigonometric functions using Euler's formulas:

$$\begin{aligned}
\cos t \sin t =& \frac{e^{it}+e^{-it}}{2}\frac{e^{it}-e^{-it}}{2i} = \frac{1}{2}\frac{e^{2it}-e^{-2it}}{2i} = \frac{1}{2}\sin(2t) \\
\int \cos t \sin t dt =& -\frac{1}{4}\cos(2t) \\
\sin^2 t =& \left(\frac{e^{it}-e^{-it}}{2i}\right)^2 = -\frac{e^{2it}+e^{-2it}-2}{4} = \frac{1}{2}-\frac{1}{2}\frac{e^{2it}+e^{-eit}}{2} = \frac{1}{2}-\frac{1}{2}\cos(2t) \\
\int \sin^2 t =& \frac{t}{2}-\frac{1}{4}\sin(2t) \\
\sin^2 t \cos t =& -\frac{\left(e^{2it}+e^{-2it}-2\right)\left(e^{it}+e^{-it}\right)}{8} = -\frac{e^{3it}+e^{it}+e^{-it}+e^{-3it}-2e^{it}-2e^{-it}}{8} \\
=& -\frac{1}{4}\left(\frac{e^{3it}+e^{-3it}}{2}-\frac{e^{it}+e^{-it}}{2}\right) = \frac{1}{4}\cos(t)-\frac{1}{4}\cos(3t) \\
\int \sin^2 t \cos t dt =& \frac{1}{4}\sin(t) - \frac{1}{12}\sin(3t)
\end{aligned}$$

Continuing our calculation,

$$\begin{aligned}
F(\gamma_2(t)) =& = (2\cos t + 2\sin t, 4\sin^2 t) \\
\gamma_2'(t) =& (-2\sin t, 2\cos t) \\
\int_{\Gamma_2} F\cdot dl =& \int_0^{2\pi} \left\langle (2\cos t + 2\sin t, 4\sin^2 t), (-2\sin t, 2\cos t)\right\rangle dt \\
=& \int_0^{2\pi} -4\cos t\sin t - 4\sin t + 8\sin^2 t\cos t dt \\
=& \left[2\cos(t) - 2\left(t - \sin t\cos t\right) + \frac{8}{3}\sin^3(t)\right]_0^{2\pi} = -4\pi
\end{aligned}$$

And finally,

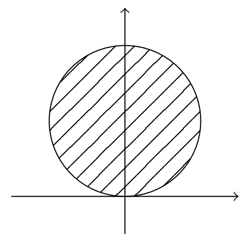
$$\int_{\partial A} F\cdot dl = -\int_{\Gamma_1} F\cdot dl + \int_{\Gamma_2} F\cdot dl = \pi - 4\pi = -3\pi$$

Exercise 2.

1. *Curl computation* : We have

$$\operatorname{rot} F(x,y) = \frac{\partial}{\partial x}\left[xy^2\right] - \frac{\partial}{\partial y}\left[-x^2 y\right] = x^2 + y^2$$

Parameterization A :



We switch to polar coordinates centered at $(0,1)$: $(x,y)=(r\cos\theta, 1+r\sin\theta)$, $r\geq 0$, $\theta\in[0,2\pi]$. Our condition becomes $x^2+(y-1)^2<1 \quad \Leftrightarrow r<1$.

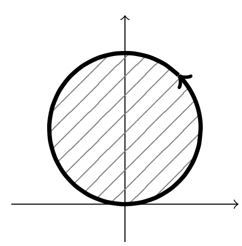
Computation of $\displaystyle\iint_A \operatorname{rot} F(x,y)dxdy$:

The Jacobian of polar coordinates is r and thus,

$$\begin{aligned}\iint_A \operatorname{rot} F(x,y)dxdy &= \int_0^1\int_0^{2\pi} \operatorname{rot} F(r\cos\theta, 1+r\sin\theta)rd\theta dr\\ &= \int_0^1\int_0^{2\pi} r^3\cos^2\theta + r\left(1+r\sin\theta\right)^2 d\theta dr\\ &= \int_0^1\int_0^{2\pi} r^3\left(\cos^2\theta+\sin^2\theta\right)+2r^2\sin\theta + rd\theta dr\\ &= \int_0^1 2\pi r^3 + 2r^2\underbrace{\left[-\cos\theta\right]_{\theta=0}^{\theta=2\pi}}_{=0} + 2\pi r dr\\ &= 2\pi\left[\frac{1}{4}r^4+\frac{1}{2}r^2\right]_0^1 = \frac{3}{2}\pi.\end{aligned}$$

Parameterization ∂A :

The boundary of A is the circle of radius 1 centered at $(0,1)$, parameterized with $\gamma\colon[0,2\pi]\to\mathbb{R}^2$ defined by $\gamma(t)=(\cos t,\sin t)$.



This parameterization leaves the domain on the left and is thus positively oriented.

Computation of $\displaystyle\int_{\partial A} F\cdot dl$:

We have,

$$\begin{aligned}
F(\gamma(t)) &= \left(-\cos^2 t - \cos^2 t \sin t, \cos t + 2\cos t \sin t + \cos t \sin^2 t\right) \\
\gamma'(t) &= (-\sin t, \cos t) \\
\int_{\partial A} F \cdot dl &= \int_0^{2\pi} \left\langle \left(-\cos^2 t - \cos^2 t \sin t, \cos t + 2\cos t \sin t + \cos t \sin^2 t\right), (-\sin t, \cos t)\right\rangle dt \\
&= \int_0^{2\pi} \cos^2 t \sin t + \cos^2 t \sin^2 t + \cos^2 t + 2\cos^2 t \sin t + \cos^2 t \sin^2 t dt \\
&= \int_0^{2\pi} 3\cos^2 t \sin t + 2\cos^2 t \sin^2 t + \cos^2 t dt
\end{aligned}$$

We find the primitive of $\cos^2 t \sin^2 t$ and $\cos^2 t$ using Euler's formulas :

$$\begin{aligned}
\cos^2 t \sin^2 t &= \left(\frac{e^{it} + e^{-it}}{2}\right)^2 \left(\frac{e^{it} - e^{-it}}{2i}\right)^2 \\
&= \frac{-1}{16}\left(\left(e^{it} + e^{-it}\right)\left(e^{it} - e^{-it}\right)\right)^2 \\
&= \frac{-1}{16}\left(e^{2it} - e^{-2it}\right)^2 \\
&= \frac{-1}{16}\left(e^{4it} - 2 + e^{-4it}\right) \\
&= \frac{1}{8} - \frac{1}{8}\frac{e^{4it} + e^{-4it}}{2} = \frac{1}{8} - \frac{1}{8}\cos(4t) \\
\int \cos^2 t \sin^2 t dt &= \frac{t}{8} - \frac{1}{32}\sin(4t) \\
\cos^2 t &= \left(\frac{e^{it} + e^{-it}}{2}\right)^2 = \frac{1}{4}\left(e^{2it} + 2 + e^{-2it}\right) \\
&= \frac{1}{2} + \frac{1}{2}\frac{e^{2it} + e^{-2it}}{2} \\
\int \cos^2 t dt &= \frac{t}{2} + \frac{1}{4}\sin(2t)
\end{aligned}$$

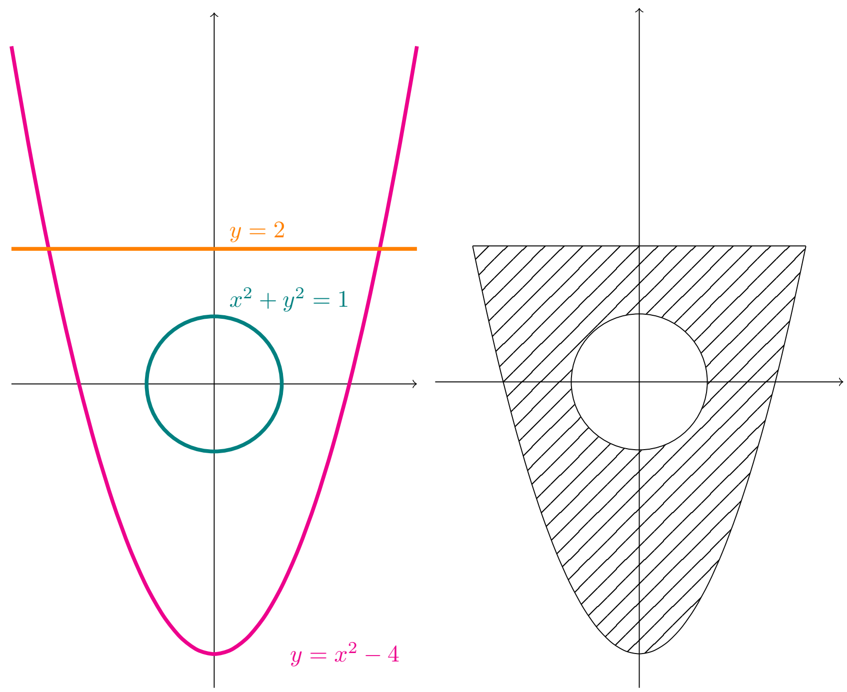
Finally,

$$\int_{\partial A} F \cdot dl = \left[-\cos^3 t + \frac{t}{4} - \frac{1}{16}\sin(4t) + \frac{t}{2} + \frac{1}{4}\sin(2t)\right]_0^{2\pi} = \frac{3\pi}{2}$$

2. *Computation of the curl of F :*

$$\operatorname{rot} F(x,y) = \frac{\partial}{\partial x}[y] - \frac{\partial}{\partial y}[xy] = -x.$$

Parameterization A and computation of $\iint_A \operatorname{rot} F(x,y) dx dy$:



To compute the integral, we therefore compute it over

$$A_0 = \left\{(x, y) \in \mathbb{R}^2 \ : \ x^2 - 4 \leq y \leq 2\right\}$$

which is y-simple and subtract the integral over the disk at the center

$$A_1 = \left\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 < 1\right\}$$

Be careful to check that the disk does not touch the other curves mathematically as well. Particularly when drawing things by hand, finding the points of intersection is important :

Intersection between $y = 2$ and $y = x^2 - 4$

$$\left\{ \begin{array}{ll} y = & 2 \\ y = & x^2 - 4 \end{array} \right. \quad \Rightarrow \quad x^2 - 4 = 2 \Rightarrow \quad x = \pm\sqrt{6}$$

and thus the two curves intersect at $(-\sqrt{6}, 2)$ and $(\sqrt{6}, 2)$.

Intersection between $y = x^2 - 4$ and $x^2 + y^2 = 1$

$$\left\{ \begin{array}{ll} y = & x^2 - 4 \\ 1 = & x^2 + y^2 \end{array} \right. \quad \Rightarrow \quad 1 = y + 4 + y^2 \quad \Rightarrow \quad y^2 + y + 3 = 0 \quad \Rightarrow y = \frac{-1 \pm \sqrt{1 - 12}}{2}$$

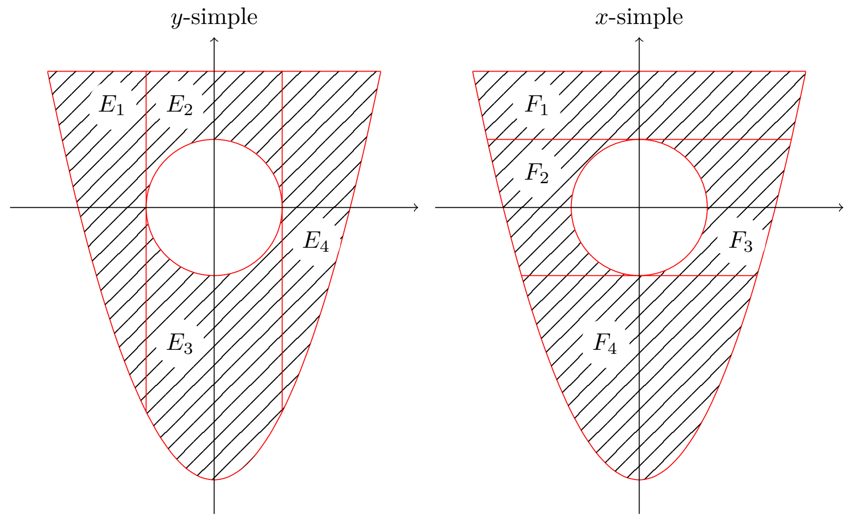
which has no solution and thus the curves do not intersect.

Intersection between $x^2+y^2=1$ and $y=2$

$$\begin{cases} 1 = & x^2+y^2 \\ y = & 2 \end{cases} \quad \Rightarrow \quad 1 = x^2+4 \quad \Rightarrow x^2=-3$$

which has no solution and thus the curves do not intersect.

An alternative to compute $\iint_A \operatorname{rot} F(x,y)dxdy = \iint_{A_0} \operatorname{rot} F(x,y)dxdy - \iint_{A_1} \operatorname{rot} F(x,y)dxdy$ is to split A into four y-simple or x-simple parts:



<u>*Variant 1*</u> : $A = A_0 \backslash A_1$

We use Cartesian coordinates for A_0 which is y-simple :

$$A_0 = \left\{(x,y)\in\mathbb{R}^2 \ : \ x\in\left[-\sqrt{6},\sqrt{6}\right], x^2-4\le y\le 2\right\}.$$

We thus have

$$\iint_{A_0} \operatorname{rot} F(x,y)dxdy = \int_{-\sqrt{6}}^{\sqrt{6}}\int_{x^2-4}^{2} -xdydx = \int_{-\sqrt{6}}^{\sqrt{6}} -6x+x^3dx = \left[-3x^2+\frac{1}{4}x^4\right]_{-\sqrt{6}}^{\sqrt{6}}$$
$$= -18+9+18-9=0$$

We use polar coordinates for A_1, $(x,y)=(r\cos\theta, r\sin\theta)$, $r\ge 0$, $\theta\in[0,2\pi]$. The condition becomes $x^2+y^2<1 \quad \Leftrightarrow r<1$. Thus,

$$\iint_{A_1} \operatorname{rot} F(x,y)dxdy == \int_0^1\int_0^{2\pi} \operatorname{rot} F(r\cos\theta, r\sin\theta)rd\theta dr$$
$$= \int_0^1\int_0^{2\pi} -r^2\cos\theta d\theta dr = -\left[\frac{1}{3}r^3\right]_0^1 [\sin\theta]_0^{2\pi} = 0$$

and finally

$$\iint_A \operatorname{rot} F(x,y)dxdy = \iint_{A_0} \operatorname{rot} F(x,y)dxdy - \iint_{A_1} \operatorname{rot} F(x,y)dxdy = 0$$

<u>*Variant 2*</u> : $A = E_1 \cup E_2 \cup E_3 \cup E_4$.

We have

$$E_1 = \left\{(x,y) \in \mathbb{R}^2 \ : \ x \in \left[-\sqrt{6}, -1\right], x^2 - 4 \le y \le 2\right\}$$

$$\begin{aligned}\iint_{E_1} \operatorname{rot} F(x,y)dxdy &= \int_{-\sqrt{6}}^{-1}\int_{x^2-4}^{2} -xdydx = \int_{-\sqrt{6}}^{-1} -6x + x^3 dx = \left[-3x^2 + \frac{1}{4}x^4\right]_{-\sqrt{6}}^{-1} \\ &= -3 + \frac{1}{4} + 18 - 9 = 6 + \frac{1}{4}\end{aligned}$$

$$E_2 = \left\{(x,y) \in \mathbb{R}^2 \ : \ x \in [-1,1], \sqrt{1-x^2} \le y \le 2\right\}$$

$$\begin{aligned}\iint_{E_2} \operatorname{rot} F(x,y)dxdy &= \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{2} -xdydx = \int_{-1}^{1} \sqrt{1-x^2}x - 2xdx = \left[-\frac{1}{3}(1-x^2)^{\frac{3}{2}} - x^2\right]_{-1}^{1} \\ &= 0 - 1 + 0 + 1 = 0\end{aligned}$$

$$E_3 = \left\{(x,y) \in \mathbb{R}^2 \ : \ x \in [-1,1], x^2 - 4 \le y \le -\sqrt{1-x^2}\right\}$$

$$\begin{aligned}\iint_{E_3} \operatorname{rot} F(x,y)dxdy &= \int_{-1}^{1}\int_{x^2-4}^{-\sqrt{1-x^2}} -xdydx = \int_{-1}^{1} \\ &= 0 - 1 + 0 + 1 = 0\end{aligned}$$

$$E_4 = \left\{(x,y) \in \mathbb{R}^2 \ : \ x \in \left[1, \sqrt{6}\right], x^2 - 4 \le y \le 2\right\}$$

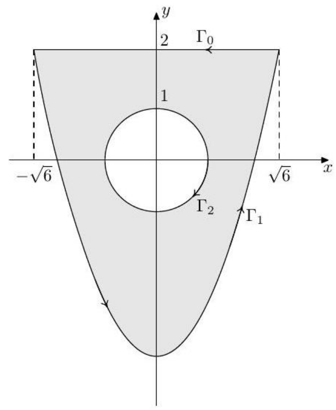
$$\begin{aligned}\iint_{E_1} \operatorname{rot} F(x,y)dxdy &= \int_{1}^{\sqrt{6}}\int_{x^2-4}^{2} -xdydx = \int_{1}^{\sqrt{6}} -6x + x^3 dx = \left[-3x^2 + \frac{1}{4}x^4\right]_{1}^{\sqrt{6}} \\ &= -18 + 9 + 3 - \frac{1}{4} = -6 - \frac{1}{4}\end{aligned}$$

$$\begin{aligned}\iint_A \operatorname{rot} F(x,y)dxdy &= \iint_{E_1} \operatorname{rot} F(x,y)dxdy + \iint_{E_2} \operatorname{rot} F(x,y)dxdy \\ &+ \iint_{E_3} \operatorname{rot} F(x,y)dxdy + \iint_{E_4} \operatorname{rot} F(x,y)dxdy = 0\end{aligned}$$

<u>*Variant 3*</u> : $A = F_1 \cup F_2 \cup F_3 \cup F_4$.

This variant is omitted.

Computation of $\int_{\partial A} F \cdot dl$.



We find that

$$\partial A = \Gamma_0 \cup \Gamma_1 \cup (-\Gamma_2)$$

where

$$\Gamma_0 = \{\alpha(t) = (-t, 2), t \in (-\sqrt{6}, \sqrt{6})\}$$
$$\Gamma_1 = \left\{\beta(t) = \left(t, t^2 - 4\right), t \in (-\sqrt{6}, \sqrt{6})\right\}$$
$$\Gamma_2 = \{\gamma(t) = (\cos t, \sin t), t \in (0, 2\pi)\}$$

The computation of the line integrals gives

$$\int_{\Gamma_0} F \cdot dl = \int_{-\sqrt{6}}^{\sqrt{6}} (-2t, 2) \cdot (-1, 0) dt = 0$$
$$\int_{\Gamma_1} F \cdot dl = \int_{-\sqrt{6}}^{\sqrt{6}} \left(t\left(t^2 - 4\right), t^2 - 4\right) \cdot (1, 2t) dt = 0$$
$$\int_{-\Gamma_2} F \cdot dl = -\int_{0}^{2\pi} (\cos t \sin t, \sin t) \cdot (-\sin t, \cos t) dt = 0$$

The final result is therefore

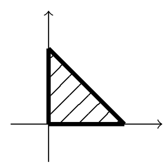
$$\int_{\partial A} F \cdot dl = \int_{\Gamma_0} F \cdot dl + \int_{\Gamma_1} F \cdot dl + \int_{-\Gamma_2} F \cdot dl = 0$$

Exercise 3.

1. We have

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) =& \frac{\partial}{\partial x}\left[y+e^x\right] = e^x \\
\frac{\partial^2 f}{\partial x^2}(x,y) =& \frac{\partial}{\partial x}\left[e^x\right] = e^x \\
\frac{\partial f}{\partial y}(x,y) =& \frac{\partial}{\partial y}\left[y+e^x\right] = 1 \\
\frac{\partial^2 f}{\partial y^2}(x,y) =& \frac{\partial}{\partial y}\left[1\right] = 0 \\
\Delta f(x,y) =& e^x.
\end{aligned}$$

The domain is either x-simple with $0 \leq x \leq 1-y$ for $y \in [0,1]$, or y-simple with $0 \leq y \leq 1-x$ for $x \in [0,1]$.



Variant 1 for the computation of $\iint_\Omega \Delta f(x,y)dxdx$: x-simple.

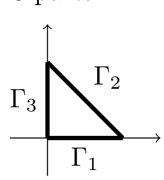
We have

$$\begin{aligned}
\iint_\Omega \Delta f(x,y)dxdy &= \int_0^1 \int_0^{1-y} e^x dxdy = \int_0^1 \left[e^x\right]_{x=0}^{x=1-y} dy = \int_0^1 e^{1-y} - 1dy \\
&= \left[-e^{1-y} - y\right]_0^1 = -1 - 1 + e + 0 = e - 2.
\end{aligned}$$

Variant 2 for the computation of $\iint_\Omega \Delta f(x,y)dxdx$: y-simple.

$$\begin{aligned}
\iint_\Omega \Delta f(x,y)dxdy &= \int_0^1 \int_0^{1-x} e^x dydx = \int_0^1 (1-x)e^x dx = \left[e^x\right]_0^1 - \int_0^1 xe^x dx \\
&\overset{\text{IBP}}{=} e - 1 - \left[xe^x\right]_0^1 + \int_0^1 e^x dx = e - 1 - e + 0 + e - 1 = e - 2
\end{aligned}$$

2. The boundary of Ω splits into 3 parts:



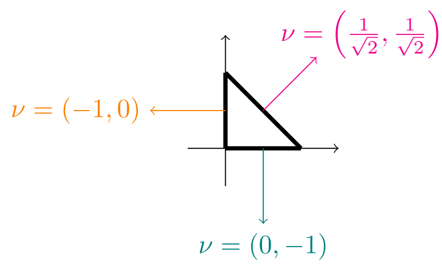
which we parametrize respectively by:

$$\begin{aligned}
\Gamma_1 : \gamma_1 \colon [0,1] \to \mathbb{R}^2 \ &: \ t \mapsto \gamma_1(t) = (t, 0) \\
\Gamma_2 : \gamma_2 \colon [0,1] \to \mathbb{R}^2 \ &: \ t \mapsto \gamma_2(t) = (t, 1-t) \\
\Gamma_3 : \gamma_3 \colon [0,1] \to \mathbb{R}^2 \ &: \ t \mapsto \gamma_3(t) = (0, t)
\end{aligned}$$

Moreover, we have

$$\nabla f(x,y) = (e^x, 1)$$

<u>*Variant 1*</u> *for the computation of* $\iint_{\partial\Omega} \langle \nabla f(x,y), \nu \rangle \, dl$: find the outward normal by inspecting the figure.



Thus,

$$\begin{aligned}
\langle \nabla f(\gamma_1(t)), \nu \rangle =& \big\langle (e^t, 1), (0, -1) \big\rangle = -1 \\
\gamma_1'(t) =& (1, 0) \\
\|\gamma_1'(t)\| =& 1 \\
\int_{\Gamma_1} \langle \nabla f(x,y), \nu \rangle \, dl =& \int_0^1 -1 \cdot 1 dt = -1
\end{aligned}$$

$$\begin{aligned}
\langle \nabla f(\gamma_2(t)), \nu \rangle =& \left\langle (e^t, 1) \, , \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\rangle = \frac{1}{\sqrt{2}} \left(e^t + 1 \right) \\
\gamma_2'(t) =& (1, -1) \\
\|\gamma_2'(t)\| =& \sqrt{2} \\
\int_{\Gamma_2} \langle \nabla f(x,y), \nu \rangle \, dl =& \int_0^1 \frac{1}{\sqrt{2}} \left(e^t + 1 \right) \sqrt{2} dt = e
\end{aligned}$$

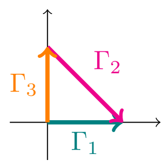
$$\begin{aligned}
\langle \nabla f(\gamma_3(t)), \nu \rangle =& \langle (1, 1), (-1, 0) \rangle = -1 \\
\gamma_3'(t) =& (0, 1) 1 \\
\int_{\Gamma_3} \langle \nabla f(x,y), \nu \rangle \, dl =& \int_0^1 -1 \cdot 1 = -1
\end{aligned}$$

Finally,

$$\int_{\partial\Omega} \langle \nabla f(x,y), \nu \rangle \, dl = \int_{\Gamma_1} \langle \nabla f(x,y), \nu \rangle \, dl + \int_{\Gamma_2} \langle \nabla f(x,y), \nu \rangle \, dl + \int_{\Gamma_3} \langle \nabla f(x,y), \nu \rangle \, dl = e-2$$

Remark.
Since $\langle \nabla f(x,y), \nu \rangle$ is scalar, there is no need to worry about the orientation of the boundary.

Variant 2 for the computation of $\iint_{\partial\Omega} \langle \nabla f(x,y), \nu \rangle \, dl$: use the formula for integrals of the type $\int_{\partial\Omega} \langle F, \nu \rangle \, dl$



We observe that Γ_1 is oriented positively while Γ_2 and Γ_3 are oriented negatively.

We have

$$\begin{aligned}
\gamma_1'(t) &= (1,0) \\
\langle \nabla f(\gamma_1(t)), ((\gamma_1')_2(t), -(\gamma_1')_1(t)) \rangle &= \big\langle (e^t, 1), (0,-1) \big\rangle = -1 \\
\int_{\Gamma_1} \langle \nabla f, \nu \rangle &= \int_0^1 -1 dt = -1
\end{aligned}$$

$$\begin{aligned}
\gamma_2'(t) &= (1,-1) \\
\langle \nabla f(\gamma_2(t)), ((\gamma_2')_2(t), -(\gamma_2')_1(t)) \rangle &= \big\langle (e^t, 1), (-1,-1) \big\rangle = -e^t - 1 \\
\int_{\Gamma_2} \langle \nabla f, \nu \rangle &= \int_0^1 -e^t - 1 dt = -e
\end{aligned}$$

$$\begin{aligned}
\gamma_3'(t) &= (0,1) \\
\langle \nabla f(\gamma_3(t)), ((\gamma_3')_2(t), -(\gamma_3')_1(t)) \rangle &= \langle (1,1), (1,0) \rangle = 1 \\
\int_{\Gamma_3} \langle \nabla f, \nu \rangle &= \int_0^1 1 dt = 1
\end{aligned}$$

Finally, taking into account the orientation of our curves:

$$\int_{\partial\Omega} \langle \nabla f(x,y), \nu \rangle \, dl = \int_{\Gamma_1} \langle \nabla f(x,y), \nu \rangle \, dl - \int_{\Gamma_2} \langle \nabla f(x,y), \nu \rangle \, dl - \int_{\Gamma_3} \langle \nabla f(x,y), \nu \rangle \, dl = e - 2$$

Exercise 4.

1. By Green's theorem, we have

$$\begin{aligned} \frac{1}{2}\int_{\partial\Omega} F \cdot dl =& \frac{1}{2}\int_{\Omega} \operatorname{rot} F(x,y) dxdy \\ =& \frac{1}{2}\int_{\Omega} \frac{\partial}{\partial x}[x] - \frac{\partial}{\partial y}[-y] \, dxdy \\ =& \frac{1}{2}\int_{\Omega} 2dxdy \\ =& \int_{\Omega} 1dxdy = \text{Area}(\Omega). \end{aligned}$$

2. By Green's theorem, we have

$$\begin{aligned} \int_{\partial\Omega} G_1 \cdot dl &= \int_{\Omega} \operatorname{rot} G_1(x,y) dxdy \\ &= \int_{\Omega} \frac{\partial}{\partial x}[x] - \frac{\partial}{\partial y}[0] \, dxdy \\ &= \int_{\Omega} 1dxdy = \text{Area}(\Omega). \end{aligned}$$

3. By Green's theorem, we have

$$\begin{aligned} \int_{\partial\Omega} G_2 \cdot dl &= \int_{\Omega} \operatorname{rot} G_2(x,y) dxdy \\ &= \int_{\Omega} \frac{\partial}{\partial x}[0] - \frac{\partial}{\partial y}[-y] \, dxdy \\ &= \int_{\Omega} 1dxdy = \text{Area}(\Omega). \end{aligned}$$