EPFL - Fall 2025	Dr. Pablo Antolin
Analysis III – MATH-202(c) – EL GM MX	**Exercises**
Correction 4	September 30 2025

Exercise 1.

Remark.
Since the domain of all our vector fields is $\mathbb{R}^2$, which is simply connected, these fields derive from a potential if and only if $\operatorname{rot} F = 0$ on $\mathbb{R}^2$. Exceptionally, in the case where F does not derive from a potential, we are asked for a closed path such that $\displaystyle\int_\Gamma F \cdot dl \neq 0$. In general, choosing a random closed curve works (if $\operatorname{rot} F \neq 0$, there are very few closed curves for which $\int_\Gamma F \cdot dl = 0$.) We will see later (with Green's theorem) that if we choose a curve which is the boundary of a region Ω such that $\operatorname{rot} F > 0$ on Ω (or $\operatorname{rot} F < 0$ on Ω), then we have

$$\int_{\partial\Omega} F \cdot dl = \int_\Omega \operatorname{rot} F dxdy > 0 \neq 0 \qquad (\text{ or } < 0 \neq 0)$$

1. We have

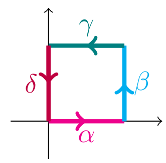
$$\operatorname{rot} F(x,y) = \frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y) = \frac{\partial}{\partial x}[xy - x] - \frac{\partial}{\partial y}[y] = x - 1 - 1 = x - 2 \neq 0$$

so F does not derive from a potential.

Let us choose Γ at random, for example $\Gamma = \left\{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\right\}$, which we parametrize with $\gamma\colon [0, 2\pi] \to \mathbb{R}^2$ defined by $\gamma(t) = (\cos t, \sin t)$. We then have

$$\begin{aligned}
F(\gamma(t)) &= (\sin t, \cos t \sin t - \cos t) \\
\gamma'(t) &= (-\sin t, \cos t) \\
\int_\Gamma F \cdot dl &= \int_0^{2\pi} \langle (\sin t, \cos t \sin t - \cos t), (-\sin t, \cos t) \rangle \, dt = \int_0^{2\pi} -\sin^2 t - \cos t \sin^2 t - \cos^2 t dt \\
&= -\int_0^{2\pi} 1 + \sin^2 t \cos t dt = \left[t + \frac{1}{3}\sin^3 t\right]_0^{2\pi} = -2\pi
\end{aligned}$$

Alternatively, we can choose the boundary of the square $[0,1] \times [0,1]$. The fact that our curve passes along the axes often gives us $x = 0$ or $y = 0$, which simplifies the calculation. Moreover, parametrizing line segments gives polynomial expressions, which we can always integrate (since the expression of F is also polynomial).



We parametrize these curves using the standard parametrization for a segment from $a = (a_1, a_2)$ to $b = (b_1, b_2)$: $t \mapsto tb + (1-t)a = a + t(b-a)$, for $t \in [0, 1]$.

$$\begin{aligned}
\alpha(t) =& t(1,0) + (1-t)(0,0) = (t,0) \\
F(\alpha(t)) =& (0, t \cdot 0 - t) = (0, -t) \\
\alpha'(t) =& (1,0) \\
\int_0^1 \langle F(\alpha(t)), \alpha'(t) \rangle \, dt =& \int_0^1 \langle (0,-t), (1,0) \rangle \, dt = \int_0^1 0 dt = 0
\end{aligned}$$

$$\begin{aligned}
\beta(t) =& t(1,1) + (1-t)(1,0) = (1,t) \\
F(\beta(t)) =& (t, t-1) \\
\beta'(t) =& (0,1) \\
\int_0^1 \langle F(\beta(t)), \beta'(t) \rangle \, dt =& \int_0^1 \langle (t, t-1), (0,1) \rangle \, dt = \int_0^1 t - 1 dt = \left[\frac{1}{2}t^2 - t\right]_0^1 = -\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\gamma(t) =& t(0,1) + (1-t)(1,1) = (1-t, 1) \\
F(\gamma(t)) =& (1, (1-t) - (1-t)) = (1, 0) \\
\gamma'(t) =& (-1, 0) \\
\int_0^1 \langle F(\gamma(t)), \gamma'(t) \rangle \, dt =& \int_0^1 \langle (-1,0), (-1,0) \rangle \, dt = \int_0^1 -1 dt = -1
\end{aligned}$$

$$\begin{aligned}
\delta(t) =& t(0,0) + (1-t)(0,1) = (0, 1-t) \\
F(\delta(t)) =& (1-t, (1-t) \cdot 0 - 0) = (1-t, 0) \\
\delta'(t) =& (0, -1) \\
\int_0^1 \langle F(\delta(t)), \delta'(t) \rangle \, dt =& \int_0^1 \langle (1-t, 0), (0,-1) \rangle \, dt = \int_0^1 0 dt = 0
\end{aligned}$$

To conclude,

$$\int_\Gamma F\cdot dl = \int_0^1 \langle F(\alpha(t)), \alpha'(t)\rangle\,dt + \int_0^1 \langle F(\beta(t)), \beta'(t)\rangle\,dt$$
$$+ \int_0^1 \langle F(\gamma(t)), \gamma'(t)\rangle\,dt + \int_0^1 \langle F(\delta(t)), \delta'(t)\rangle\,dt = -\frac{3}{2} \neq 0$$

2. We have

$$\operatorname{rot} F(x,y) = \frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y) = \frac{\partial}{\partial x}\left[x^3\right] - \frac{\partial}{\partial y}\left[3x^2y + 2x\right] = 3x^2 - 3x^2 = 0$$

and therefore F derives from a potential. We must then find $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x}(x,y) = F_1(x,y) = 3x^2y + 2x$$
$$\frac{\partial f}{\partial y}(x,y) = F_2(x,y) = x^3$$

We can choose to either start by integrating the first equation with respect to x or start by integrating the second equation with respect to y.

Option 1: Start by integrating the first equation with respect to x.

We have

$$f(x,y) = \int \frac{\partial f}{\partial x}(x,y)dx = \int 3x^2y + 2x dx = x^3y + x^2 + C(y),$$

where, since we integrated with respect to x, our constant may depend on y. We substitute this into our second equation:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^3y + x^2 + C(y)\right] = x^3 + C'(y) := x^3.$$

Thus, we must have $C'(y) = 0$, meaning that C is a constant. Finally, the potentials of F are

$$f(x,y) = x^3y + x^2 + C,$$

with $C \in \mathbb{R}$. (There is no need to keep the constant; giving one potential is enough.)

Option 2: Start by integrating the second equation with respect to y.

We have

$$f(x,y) = \int \frac{\partial f}{\partial y}(x,y)dy = \int x^3 dy = x^3y + C(x),$$

where, since we integrated with respect to y, our constant may depend on x. We substitute this into our first equation:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^3y + C(x)\right] = 3x^2y + C'(x) := 3x^2y + 2x.$$

Thus, we must have $C'(x) = 2x$, so $C(x) = x^2 + K$ where K is a constant. Finally, the potentials of F are

$$f(x, y) = x^3y + x^2 + K,$$

with $K \in \mathbb{R}$. (There is no need to keep the constant; giving one potential is enough.)

3. We have

$$\operatorname{rot} F(x,y) = \frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y) = \frac{\partial}{\partial x}\left[x^2\right] - \frac{\partial}{\partial y}\left[3x^2y\right] = 2x - 3x^2 \neq 0.$$

so F does not derive from a potential.

Let us choose Γ at random, for example $\Gamma = \left\{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\right\}$ parametrized with $\gamma \colon [0, 2\pi] \to \mathbb{R}^2$ defined by $\gamma(t) = (\cos(t), \sin(t))$. We then have

$$\begin{aligned} F(\gamma(t)) &= \left(3\cos^2 t \sin t, \cos^2 t\right) \\ \gamma'(t) &= (-\sin t, \cos t) \\ \int_\Gamma F \cdot dl &= \int_0^{2\pi} \left\langle \left(3\cos^2 t \sin t, \cos^2 t\right), (-\sin t, \cos t)\right\rangle dt = \int_0^{2\pi} -3\cos^2 t \sin^2 t + \cos^3 t dt. \end{aligned}$$

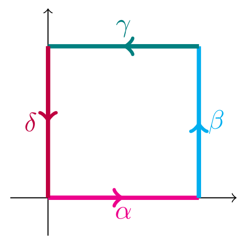
We compute the antiderivatives using the technique developed in Series 1 with Euler's formulas:

$$\begin{aligned} \cos^2 t \sin^2 t &= \left(\frac{e^{it} + e^{-it}}{2}\right)^2 \left(\frac{e^{it} - e^{-it}}{2i}\right)^2 = \frac{-1}{16}\left(e^{2it} + 2 + e^{-2it}\right)\left(e^{2it} - 2 + e^{-2it}\right) \\ &= \frac{-1}{16}\left(e^{4it} - 2e^{2it} + 1 + 2e^{2it} - 4 + 2e^{-2it} + 1 - e^{-2it} + e^{-4it}\right) \\ &= \frac{-1}{16}\left(e^{4it} + e^{4it} - 2\right) = -\frac{1}{8}(\cos 4t - 1) \\ \cos^3 t &= \left(\frac{e^{it} + e^{-it}}{2}\right)^3 = \frac{1}{8}\left(e^{3it} + e^{-3it} + 3e^{it} + 3e^{-it}\right) \\ &= \frac{1}{4}\cos(3t) + \frac{3}{4}\cos(t), \end{aligned}$$

thus,

$$\begin{aligned} \int_\Gamma F \cdot dl &= \int_0^{2\pi} \frac{3}{8}\cos 4t + \frac{3}{8} + \frac{1}{4}\cos(3t) + \frac{3}{4}\cos(t) dt \\ &= \left[\frac{3}{32}\sin(4t) + \frac{3}{8}t + \frac{1}{12}\sin(3t) + \frac{3}{4}\sin(t)\right]_0^{2\pi} = \frac{3\pi}{4} \neq 0. \end{aligned}$$

Alternatively, as in point 1, we could take the boundary of the square $[0,2] \times [0,2]$ (if we take the boundary of the square $[0,1] \times [0,1]$, we unfortunately get 0. Bad luck...)



We then have,

$$\begin{aligned} \alpha(t) =& (2t, 0) \\ F(\alpha(t)) =& (3(2t)^2 \cdot 0, (2t)^2) = (0, 4t^2) \\ \alpha'(t) =& (2, 0) \\ \int_0^1 \langle F(\alpha(t)), \alpha'(t) \rangle \, dt =& \int_0^1 \left\langle (0, 4t^2), (1, 0) \right\rangle dt = \int_0^1 0 dt = 0 \end{aligned}$$

$$\begin{aligned} \beta(t) =& (2, 2t) \\ F(\beta(t)) =& (24t, 4) \\ \beta'(t) =& (0, 2) \\ \int_0^1 \langle F(\beta(t)), \beta'(t) \rangle \, dt =& \int_0^1 \langle (24t, 4), (0, 2) \rangle \, dt = \int_0^1 8 dt = 8 \end{aligned}$$

$$\begin{aligned} \gamma(t) =& (2 - 2t, 2) \\ F(\gamma(t)) =& (24(1-t)^2, 4(1-t)^2) \\ \gamma'(t) =& (-2, 0) \\ \int_0^1 \langle F(\gamma(t)), \gamma'(t) \rangle \, dt =& \int_0^1 \left\langle (24(1-t)^2, 4(1-t)^2), (-2, 0) \right\rangle dt \\ =& \int_0^1 -48(1-t)^2 dt = \left[16(1-t)^3\right]_0^1 = -16 \end{aligned}$$

$$\begin{aligned} \delta(t) =& (0, 2 - 2t) \\ F(\delta(t)) =& (3 \cdot 0 \cdot (2 - 2t), 0) = (0, 0) \\ \delta'(t) =& (0, -2) \\ \int_0^1 \langle F(\delta(t)), \delta'(t) \rangle \, dt =& \int_0^1 \langle (0, 0), (0, -2) \rangle \, dt = \int_0^1 0 dt = 0. \end{aligned}$$

Finally,

$$\int_\Gamma F \cdot dl = \int_0^1 \langle F(\alpha(t)), \alpha'(t)\rangle \, dt + \int_0^1 \langle F(\beta(t)), \beta'(t)\rangle \, dt + \int_0^1 \langle F(\gamma(t)), \gamma'(t)\rangle \, dt + \int_0^1 \langle F(\delta(t)), \delta'(t)\rangle \, dt = -8 \neq 0$$

Exercise 2.
We have

$$\begin{aligned}
\frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}\left[2xy + \frac{z}{1+x^2}\right] = 2x \\
\frac{\partial F_1}{\partial z} &= \frac{\partial}{\partial z}\left[2xy + \frac{z}{1+x^2}\right] = \frac{1}{1+x^2} \\
\frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}\left[x^2 + 2yz\right] = 2x \\
\frac{\partial F_2}{\partial z} &= \frac{\partial}{\partial z}\left[x^2 + 2yz\right] = 2y \\
\frac{\partial F_3}{\partial x} &= \frac{\partial}{\partial x}\left[y^2 + \arctan x\right] = \frac{1}{1+x^2} \\
\frac{\partial F_3}{\partial y} &= \frac{\partial}{\partial y}\left[y^2 + \arctan x\right] = 2y
\end{aligned}$$

$$\operatorname{rot} F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = (0,0,0)$$

Moreover, since the domain of definition of F is $\mathbb{R}^3$, which is simply connected, F derives from a potential. We therefore seek f such that

$$\begin{cases}
\dfrac{\partial f}{\partial x} = F_1 = 2xy + \dfrac{z}{1+x^2} \\
\dfrac{\partial f}{\partial y} = F_2 = x^2 + 2yz \\
\dfrac{\partial f}{\partial z} = F_3 = y^2 + \arctan x
\end{cases}$$

Option 1: Integrate the first equation with respect to x.

We have

$$f(x,y,z) = \int \frac{\partial f}{\partial x}(x,y,z)dx = \int 2xy + \frac{z}{1+x^2}dx = x^2y + z\arctan x + C(y,z),$$

where, since we integrated with respect to x, the constant may depend on y and z.

Option 1.1: Substitute into the second equation and integrate with respect to y.

We have

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y,z) =& \frac{\partial}{\partial y}\left[x^2y + z\arctan x + C(y,z)\right] = x^2 + \frac{\partial C}{\partial y}(y,z) := x^2 + 2yz\\ \Rightarrow \frac{\partial C}{\partial y}(y,z) =& 2yz\\ C(y,z) =& \int \frac{\partial C}{\partial y}(y,z)dy = \int 2yzdy = y^2z + K(z),\\ \Rightarrow f(x,y,z) =& x^2y + y^2z + z\arctan x + K(z)\end{aligned}$$

where, since we integrated with respect to y, the constant K may depend on z (but not on x since C does not depend on x).

Finally, substituting this into the third equation and integrating with respect to z, we have

$$\begin{aligned}\frac{\partial f}{\partial z}(x,y,z) =& \frac{\partial}{\partial z}\left[x^2y + y^2z + z\arctan x + K(z)\right] = y^2 + \arctan x + K'(z) := y^2 + \arctan x\\ \Rightarrow K'(z) =& 0\\ K(z) =& \text{const}\end{aligned}$$

and finally,

$$f(x,y,z) = x^2y + y^2z + z\arctan x + K,$$

with $K \in \mathbb{R}$. (No need to keep the constant; giving one potential is enough.)

Option 1.2: Substitute into the third equation and integrate with respect to z.

We have

$$\begin{aligned}\frac{\partial f}{\partial z}(x,y,z) =& \frac{\partial}{\partial z}\left[x^2y + z\arctan x + C(y,z)\right] = \arctan x + \frac{\partial C}{\partial z}(y,z) := y^2 + \arctan x\\ \Rightarrow \frac{\partial C}{\partial z}(y,z) =& y^2\\ C(y,z) =& \int \frac{\partial C}{\partial z}(y,z)dz = y^2z + K(y)\\ \Rightarrow f(x,y,z) =& x^2y + y^2z + z\arctan x + K(y)\end{aligned}$$

where, since we integrated with respect to z, the constant K may depend on y (but not on x since C does not depend on x).

Finally, substituting this into the second equation and integrating with respect to y, we have

$$\begin{aligned}
\frac{\partial f}{\partial y}(x,y,z) =& \frac{\partial}{\partial y}\left[x^2y + y^2z + z\arctan x + K(y)\right] = x^2 + 2yz + K'(y) := x^2 + 2yz \\
\Rightarrow K'(y) =& 0 \\
K(y) =& \text{const}
\end{aligned}$$

and finally

$$f(x,y,z) = x^2y + y^2z + z\arctan x + K,$$

with $K \in \mathbb{R}$. (No need to keep the constant; giving one potential is enough.)

Option 2: Integrate the second equation with respect to y.

We have

$$f(x,y,z) = \int \frac{\partial f}{\partial y}(x,y,z)dy = \int x^2 + 2yz\,dy = x^2y + y^2z + C(x,z)$$

where, since we integrated with respect to y, the constant may depend on x and z.

Option 2.1: Substitute into the first equation and integrate with respect to x.

We have

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) =& \frac{\partial}{\partial x}\left[x^2y + y^2z + C(x,z)\right] = 2xy + \frac{\partial C}{\partial x}(x,z) := 2xy + \frac{z}{1+x^2} \\
\Rightarrow \frac{\partial C}{\partial x}(x,z) =& \frac{z}{1+x^2} \\
C(x,z) =& \int \frac{\partial C}{\partial x}(x,z)dx = \int \frac{z}{1+x^2}dx = z\arctan x + K(z) \\
\Rightarrow f(x,y,z) =& x^2y + y^2z + z\arctan x + K(z)
\end{aligned}$$

where, since we integrated with respect to x, the constant K may depend on z (but not on y since C does not depend on y).

Finally, substituting this into the third equation and integrating with respect to z, we have

$$\begin{aligned}
\frac{\partial f}{\partial z}(x,y,z) =& \frac{\partial}{\partial z}\left[x^2y + y^2z + z\arctan x + K(z)\right] = y^2 + \arctan x + K'(z) := y^2 + \arctan x \\
\Rightarrow K'(z) =& 0 \\
K(z) =& \text{const}
\end{aligned}$$

and finally,

$$f(x,y,z) = x^2y + y^2z + z\arctan x + K,$$

with $K \in \mathbb{R}$. (No need to keep the constant; giving one potential is enough.)

Option 2.2: Substitute into the third equation and integrate with respect to z.

We have

$$
\begin{aligned}
\frac{\partial f}{\partial z}(x,y,z) =& \frac{\partial}{\partial z}\left[x^2y + y^2z + C(x,z)\right] = y^2 + \frac{\partial C}{\partial z}(x,z) := y^2 + \arctan x \\
\Rightarrow \frac{\partial C}{\partial z}(x,z) =& \arctan x \\
C(x,z) =& \int \frac{\partial C}{\partial z}(x,z)dz = \int \arctan x dz = z\arctan x + K(x) \\
\Rightarrow f(x,y,z) =& x^2y + y^2z + z\arctan x + K(x)
\end{aligned}
$$

where, since we integrated with respect to z, the constant K may depend on x (but not on y since C does not depend on y).

Finally, substituting this into the first equation and integrating with respect to x, we have

$$
\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) =& \frac{\partial}{\partial x}\left[x^2y + y^2z + z\arctan x + K(x)\right] = 2xy + \frac{z}{1+x^2} := 2xy + \frac{z}{1+x^2} \\
\Rightarrow K'(x) =& 0 \\
K(x) =& \text{const}
\end{aligned}
$$

and finally,

$$
f(x,y,z) = x^2y + y^2z + z\arctan x + K,
$$

with $K \in \mathbb{R}$. (No need to keep the constant; giving one potential is enough.)

Option 3: Integrate the third equation with respect to z.

We have

$$
f(x,y,z) = \int \frac{\partial f}{\partial z}(x,y,z)dz = \int y^2 + \arctan x dz = y^2z + z\arctan x + C(x,y),
$$

where, since we integrated with respect to z, the constant C may depend on x and y.

Option 3.1: Substitute into the first equation and integrate with respect to x.

We have

$$
\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) =& \frac{\partial}{\partial x}\left[y^2z + z\arctan x + C(x,y)\right] = \frac{z}{1+x^2} + \frac{\partial C}{\partial x}(x,y) := 2xy + \frac{z}{1+x^2} \\
\Rightarrow \frac{\partial C}{\partial x}(x,y) =& 2xy \\
C(x,y) =& \int \frac{\partial C}{\partial x}(x,y)dx = \int 2xydx = x^2y + K(y) \\
\Rightarrow f(x,y,z) =& x^2y + y^2z + z\arctan x + K(y)
\end{aligned}
$$

where, since we integrated with respect to x, the constant K may depend on y (but not on z since C does not depend on z).

Finally, substituting this into the second equation and integrating with respect to y, we have

$$\begin{aligned}
\frac{\partial f}{\partial y}(x,y,z) =& \frac{\partial}{\partial y}\left[x^2y + y^2z + z\arctan x + K(y)\right] = x^2 + 2yz + K'(y) := x^2 + 2yz \\
\Rightarrow K'(y) =& 0 \\
K(y) =& \text{const}
\end{aligned}$$

and finally

$$f(x,y,z) = x^2y + y^2z + z\arctan x + K,$$

with $K \in \mathbb{R}$. (No need to keep the constant; giving one potential is enough.)

Option 3.2: Substitute into the second equation and integrate with respect to y.

We have

$$\begin{aligned}
\frac{\partial f}{\partial y}(x,y,z) =& \frac{\partial}{\partial y}\left[y^2z + z\arctan x + C(x,y)\right] = 2yz + \frac{\partial C}{\partial y}(x,y) := x^2 + 2yz \\
\Rightarrow \frac{\partial C}{\partial y}(x,y) =& x^2 \\
C(x,y) =& \int \frac{\partial C}{\partial y}(x,y)dy = \int x^2 dy = x^2y + K(x) \\
\Rightarrow f(x,y,z) =& x^2y + y^2z + z\arctan x + K(x)
\end{aligned}$$

where, since we integrated with respect to y, the constant K may depend on x (but not on z since C does not depend on z).

Finally, substituting this into the first equation and integrating with respect to x, we have

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) =& \frac{\partial}{\partial x}\left[x^2y + y^2z + \frac{z}{1+x^2} + K(x)\right] = 2xy + \frac{z}{1+x^2} := 2xy + \frac{z}{1+x^2} \\
\Rightarrow K'(x) =& 0 \\
K(x) =& \text{const}
\end{aligned}$$

and finally,

$$f(x,y,z) = x^2y + y^2z + z\arctan x + K,$$

with $K \in \mathbb{R}$. (No need to keep the constant; giving one potential is enough.)

Exercise 3.
For the vector fields $F\colon \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}^2$ below, do they derive from a potential? If yes, give a potential; if not, justify your answer.

1. $F(x,y) = \left(\frac{-x}{(x^2+y^2)^2}, \frac{-y}{(x^2+y^2)^2}\right)$

2. $F(x,y) = \left(\dfrac{y^3}{(x^2+y^2)^2}, \dfrac{-xy^2}{(x^2+y^2)^2}\right)$

We use the method from the course. **In this solution, whenever we have to choose between looking for a potential and looking for a closed path, we will systematically make the wrong choice to see how to detect that we went in the wrong direction.**

1. *Step 1*: Compute $\operatorname{rot} F$.

   We have

$$\begin{aligned}\operatorname{rot} F(x,y) =&\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y) = \frac{\partial}{\partial x}\left[\frac{-y}{(x^2+y^2)^2}\right] - \frac{\partial}{\partial y}\left[\frac{-x}{(x^2+y^2)^2}\right] \\ =&\left(-y(-2)\frac{\frac{\partial}{\partial x}\left[x^2+y^2\right]}{(x^2+y^2)^3}\right) - \left(-x(-2)\frac{\frac{\partial}{\partial y}\left[x^2+y^2\right]}{(x^2+y^2)^3}\right) \\ =&4y\frac{x}{(x^2+y^2)^3} - \left(4x\frac{y}{(x^2+y^2)^3}\right) = 0\end{aligned}$$

   The curl of F is zero, so we move to

   Step 2: Domain of F.

   The domain of F is $\mathbb{R}^2\backslash\{(0,0)\}$, which is not simply connected; it has a hole at the origin. So we move to step 3, for which we have a choice: try to find a potential, or try to compute the integral over a closed path that surrounds the hole in the domain. Here, the field will derive from a potential (there's no real way to know this at this stage); for pedagogical reasons, we first test trying to find a curve that surrounds a hole in the domain.

   Step 3: For each hole in the domain, integrate F over a closed path that surrounds that hole.

   Here, the only hole in the domain is the origin, so we only have one path to find, and we choose $\Gamma = \left\{(x,y)\in\mathbb{R}^2 \;:\; x^2+y^2=1\right\}$, parameterized by $\gamma\colon [0,2\pi]\to\mathbb{R}^2$ defined by $\gamma(t) = (\cos t, \sin t)$.

   We then have

$$\begin{aligned}F(\gamma(t)) =&(-\cos t, -\sin t) \\ \gamma'(t) =&(-\sin t, \cos t)\end{aligned}$$

$$\int_\gamma F\cdot dl = \int_0^{2\pi} \langle(-\cos t, -\sin t), (-\sin t, \cos t)\rangle\, dt = \int_0^{2\pi} \cos t\sin t - \cos t\sin t\, dt = \int_0^{2\pi} 0\, dt = 0$$

   Since there is only one hole in the domain and the integral over a closed contour around this hole gives 0, we suspect that the integral over all closed paths is zero, i.e., that F derives from a potential. We therefore change our approach: we will look for a potential for F.

Step 4: We look for a potential.

We seek $f \in C^1(\mathbb{R}^2 \backslash \{(0,0)\})$ such that

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = & \dfrac{-x}{(x^2+y^2)^2} \\ \\ \dfrac{\partial f}{\partial y}(x,y) = & \dfrac{-y}{(x^2+y^2)^2} \end{cases}$$

Since our expressions are symmetric in x and y, we choose to start by integrating the first equation with respect to x. The calculations in the case where we start by integrating the second equation with respect to y are the same; we just swap the roles of x and y.

We have

$$f(x,y) = \int \frac{\partial f}{\partial x}(x,y)dx = \int \frac{-x}{(x^2+y^2)^2}dx = \frac{1}{2(x^2+y^2)} + C(y),$$

where, since we integrated with respect to x, the constant is allowed to depend on y.

Injecting into the second equation gives

$$\begin{aligned} \frac{\partial f}{\partial y}(x,y) =& \frac{\partial}{\partial y}\left[\frac{1}{2(x^2+y^2)} + C(y)\right] = \frac{-y}{2(x^2+y^2)^2} + C'(y) := \frac{-y}{2(x^2+y^2)^2} \\ \Rightarrow C'(y) =& 0 \\ C(y) =& const. \end{aligned}$$

Finally, the potentials of F are

$$f(x,y) = \frac{1}{2(x^2+y^2)} + C,$$

with $C \in \mathbb{R}$.

2. *Step 1*: Compute $\operatorname{rot} F$.

   We have

$$
\begin{aligned}
\operatorname{rot} F(x,y) =& \frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y) = \frac{\partial}{\partial x}\left[\frac{-xy^2}{(x^2+y^2)^2}\right] - \frac{\partial}{\partial y}\left[\frac{y^3}{(x^2+y^2)^2}\right] \\
=& \frac{-y^2(x^2+y^2)^2 + xy^2\frac{\partial}{\partial x}\left[(x^2+y^2)^2\right]}{(x^2+y^2)^4} - \frac{3y^2(x^2+y^2)^2 - y^3\frac{\partial}{\partial y}\left[(x^2+y^2)^2\right]}{(x^2+y^2)^4} \\
=& \frac{-y^2(x^2+y^2)^2 + xy^2\cdot 4x(x^2+y^2)}{(x^2+y^2)^4} - \frac{3y^2(x^2+y^2)^2 - y^3\cdot 4y(x^2+y^2)}{(x^2+y^2)^4} \\
=& \frac{-y^2(x^2+y^2) + 4x^2y^2 - 3y^2(x^2+y^2) + 4y^4}{(x^2+y^2)^3} \\
=& \frac{-y^2x^2 - y^4 + 4x^2y^2 - 3y^2x^2 - 3y^4 + 4y^4}{(x^2+y^2)^3} = 0
\end{aligned}
$$

The curl of F is zero, so we move to

Step 2: Domain of F.

The domain of F is $\mathbb{R}^2\backslash\{(0,0)\}$, which is not simply connected; it has a hole at the origin. We therefore move to Step 3, where we have a choice: either try to find a potential, or try to compute the integral over a closed path that surrounds the hole in the domain. Here, the field will not derive from a potential (there is no real way to know this at this stage), but for pedagogical reasons we first attempt to find a potential.

Step 3: We look for a potential for F.

We seek $f \in C^1(\mathbb{R}^2\backslash\{(0,0)\})$ such that

$$
\begin{cases}
\dfrac{\partial f}{\partial x}(x,y) = & \dfrac{y^3}{(x^2+y^2)^2} \\[2ex]
\dfrac{\partial f}{\partial y}(x,y) = & \dfrac{-xy^2}{(x^2+y^2)^2}
\end{cases}
$$

We integrate the first equation with respect to x. (We could integrate the second equation with respect to y; the calculations are not significantly easier, but not trivial either.)

We have

$$
f(x,y) = \int \frac{\partial f}{\partial x}(x,y)dx = \int \frac{y^3}{(x^2+y^2)^2}dx = \int \frac{y^{-4}}{y^{-4}}\frac{y^3}{(x^2+y^2)^2}dx = \int \frac{1}{y}\frac{1}{\left(1+\frac{x^2}{y^2}\right)^2}dx.
$$

At this stage, it is not unreasonable that the awkward form of our integral makes us reconsider and either try integrating the second equation or find a closed path over which the integral of F does not vanish.

Indeed, the antiderivative of $\frac{1}{(1+t^2)^2}$ is not part of the standard Analysis I curriculum. However, this antiderivative is useful in the generalization of the method of decomposition into partial fractions when, in the factorization of the denominator, we have terms of the form $(x^2+bx+c)^2$ where the polynomial x^2+bx+c is irreducible, i.e., it has no real roots.

Here we propose a method to calculate it (this **is not** the kind of thing you will need to know for the exam).

$$
\begin{aligned}
\int \frac{1}{(1+x^2)^2}dx = \int \frac{1+x^2}{(1+x^2)^2} - \frac{x^2}{(1+x^2)^2}dx = \int \frac{1}{1+x^2}dx + \int \frac{x}{2}\frac{-2x}{(1+x^2)^2}dx \\
\overset{\text{IBP}}{=} \arctan(x) + \frac{x}{2}\frac{1}{1+x^2} - \int \frac{1}{2}\frac{1}{1+x^2}dx \\
=\frac{1}{2}\frac{x}{1+x^2} + \arctan(x) - \frac{1}{2}\arctan(x) \\
=\frac{1}{2}\left(\frac{x}{1+x^2} + \arctan(x)\right).
\end{aligned}
$$

Continuing our calculation,

$$
\begin{aligned}
f(x,y) = \int \frac{1}{y}\frac{1}{\left(1+\left(\frac{x}{y}\right)^2\right)^2}dx = \frac{1}{2}\left(\frac{\frac{x}{y}}{1+\frac{x^2}{y^2}} + \arctan\left(\frac{x}{y}\right)\right) + C(y) \\
=\frac{1}{2}\frac{xy}{x^2+y^2} + \frac{1}{2}\arctan\left(\frac{x}{y}\right) + C(y).
\end{aligned}
$$

At this stage, notice that the term $\arctan\left(\frac{x}{y}\right)$ does not have the same domain as F. Even without knowing $C(y)$ yet, the fact that $C(y)$ cannot depend on x, while $\arctan\left(\frac{x}{y}\right)$ depends on x, indicates that it is very unlikely that the exact expression of $C(y)$ could "fix" the domain issue with $\arctan(x/y)$. Specifically, this means that if, for example, $C(y) = -\frac{1}{2}\arctan\left(\frac{x}{y}\right)$ were admissible, then f would have no domain problem. But such a function $C(y)$ is not allowed because $\arctan(x/y)$ depends on x, which is not permitted for $C(y)$.

This is a hint that we have probably taken the wrong approach. But let's persevere, just out of morbid curiosity.

We inject this into the second equation:

$$\begin{aligned}
\frac{\partial f}{\partial y}(x,y) =& \frac{\partial}{\partial y}\left[\frac{1}{2}\frac{xy}{x^2+y^2} + \frac{1}{2}\arctan\left(\frac{x}{y}\right) + C(y)\right] \\
=& \frac{1}{2}\frac{x(x^2+y^2) - 2y\cdot xy}{(x^2+y^2)^2} + \frac{1}{2}\frac{1}{1+\frac{x^2}{y^2}}\frac{-x}{y^2} + C'(y) \\
=& \frac{1}{2}\frac{x^3+xy^2-2xy^2}{(x^2+y^2)^2} - \frac{1}{2}\frac{x^3+xy^2}{(x^2+y^2)^2} + C'(y) \\
=& \frac{-xy^2}{(x^2+y^2)^2} + C'(y) \\
\Rightarrow C'(y) =& 0 \\
C(y) =& const.
\end{aligned}$$

Finally,

$$f(x,y) = \frac{1}{2}\frac{xy}{x^2+y^2} + \frac{1}{2}\arctan\left(\frac{x}{y}\right) + C,$$

with $C \in \mathbb{R}$.

As already mentioned, we cannot choose a constant such that $f \in C^1(\mathbb{R}^2\backslash\{(0,0)\})$, so we change approach and look for a closed curve on which the integral does not vanish.

Step 4: For each hole in the domain, we look for a closed curve surrounding the hole such that the integral of F over this curve does not vanish.

Since $(0,0)$ is the only hole in the domain, we choose $\Gamma = \left\{(x,y) \in \mathbb{R}^2 \;:\; x^2+y^2 = 1\right\}$ parametrized by $\gamma\colon [0,2\pi] \to \mathbb{R}^2$ defined by $\gamma(t) = (\cos t, \sin t)$.

We then have

$$\begin{aligned}
F(\gamma(t)) =& (\sin^3 t, -\cos t \sin^2 t) \\
\gamma'(t) =& (-\sin t, \cos t) \\
\int_\Gamma F \cdot dl =& \int_0^{2\pi} \langle F(\gamma(t)), \gamma'(t)\rangle \, dt = \int_0^{2\pi} -\sin^4 t - \cos^2 t \sin^2 t dt \\
=& \int_0^{2\pi} -\sin^2 t\left(\sin^2 t + \cos^2 t\right) dt = -\int_0^{2\pi} \sin^2 t dt.
\end{aligned}$$

We find an antiderivative of $\sin^2 t$ using Euler's formula:

$$\sin^2 t = \left(\frac{e^{it} - e^{-it}}{2i}\right)^2 = -\frac{1}{4}\left(e^{2it} - 2 + e^{-2it}\right) = \frac{1}{2} - \frac{1}{2}\cos(2t)$$

and therefore,

$$\int_\Gamma F \cdot dl = \int_0^{2\pi} \frac{1}{2}\cos(2t) - \frac{1}{2} dt = \left[\frac{1}{4}\sin(2t) - \frac{t}{2}\right]_0^{2\pi} = -\pi \neq 0$$

and thus F does not derive from a potential.

Exercise 4.

1. Given that the vector field $F(x,y) = (F_1(x,y), F_2(x,y))$ derives from a potential f in $\mathbb{R}^2$, we know:

$$F_1(x,y) = \frac{\partial f(x,y)}{\partial x} \quad \text{and} \quad F_2(x,y) = \frac{\partial f(x,y)}{\partial y},$$

or equivalently

$$F_1(t,u(t)) = \frac{\partial f(t,u(t))}{\partial t} \quad \text{and} \quad F_2(t,u(t)) = \frac{\partial f(t,u(t))}{\partial u}.$$

Notice that the second derivative (respect to u) is just a partial derivative of F_2 respect its second variable.

To show that $f(t,u(t))$ is constant, we need to calculate its total derivative with respect to t:

$$\frac{d}{dt} f(t,u(t)) = \frac{\partial f}{\partial t}(t,u(t)) + u'(t)\frac{\partial f}{\partial u}(t,u(t)).$$

Here, we substitute:

$$\frac{d}{dt} f(t,u(t)) = F_1(t,u(t)) + F_2(t,u(t))u'(t).$$

From the differential equation:

$$F_2(t,u(t))u'(t) + F_1(t,u(t)) = 0,$$

therefore,

$$\frac{d}{dt} f(t,u(t)) = 0.$$

Since the derivative is equal to zero, $f(t,u(t))$ is constant for all $t \in \mathbb{R}$.

Thus, we have shown that a solution $u(t)$, in implicit form, of the differential equation is given by:

$$f(t,u(t)) = \text{constant}$$

for all $t \in \mathbb{R}$.

2. Given the equation:

$$u^2(t)u'(t) + \sin t = 0,$$

we can rewrite it as follows

$$F_2(t,u(t))u'(t) + F_1(t,u(t)) = 0,$$

where $F_1 = \sin t$ and $F_2 = u^2(t)$.

Since

$$\frac{d}{dt}f(t,u(t)) = F_1(t,u(t)) + F_2(t,u(t))u'(t),$$

and the fact that F is defined for $t \in \mathbb{R}$ we can compute f by taking an integral:

$$f(t,u(t)) = -\cos t + \frac{u^3(t)}{3}.$$

We just have shown that shown that a solution $u(t)$, in implicit form, of the differential equation is given by:

$$f(t,u(t)) = \text{constant}.$$

Therefore

$$-\cos t + \frac{u^3(t)}{3} = c = \text{constant}.$$

Given the initial condition $u(0) = 3$, we can find the constant c:

$$-\cos 0 + \frac{u^3(0)}{3} = -1 + 9 = 8 = c.$$

Finally, combining the two last equations we get the solution:

$$u(t) = (3\cos t + 24)^{1/3}.$$

Exercise 5.

For $i = 1, 2$, we have

$$\begin{aligned}\frac{d}{dt}\left[tF_i(tx,ty)\right] =& F_1(tx,ty) + t\left(\frac{\partial F_i}{\partial x}(tx,ty)\frac{d}{dt}\left[tx\right] + \frac{\partial F_i}{\partial y}(tx,ty)\frac{d}{dt}\left[ty\right]\right)\\ =& F_1(tx,ty) + tx\frac{\partial F_i}{\partial x}(tx,ty) + ty\frac{\partial F_i}{\partial y}(tx,ty).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{\partial \varphi}{\partial x}(x,y) =& \frac{\partial}{\partial x}\left[\int_0^1 xF_1(tx,ty) + yF_2(tx,ty)dt\right]\\ =& \int_0^1 \frac{\partial}{\partial x}\left[xF_1(tx,ty) + yF_2(tx,ty)\right]dt\\ =& \int_0^1 F_1(tx,ty) + x\left(\frac{\partial F_1}{\partial x}(tx,ty)\frac{\partial}{\partial x}\left[tx\right] + \frac{\partial F_1}{\partial y}(tx,ty)\frac{\partial}{\partial x}\left[ty\right]\right)\\ &+ y\left(\frac{\partial F_2}{\partial x}(tx,ty)\frac{\partial}{\partial x}\left[tx\right] + \frac{\partial F_2}{\partial y}(tx,ty)\frac{\partial}{\partial x}\left[ty\right]\right)dt\\ =& \int_0^1 F_1(tx,ty) + x\frac{\partial F_1}{\partial x}(tx,ty)t + y\frac{\partial F_2}{\partial x}(ty,ty)tdt\end{aligned}$$

Using that $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ and the calculation above, we obtain

$$\begin{aligned}\frac{\partial \varphi}{\partial x}(x,y) &= \int_0^1 F_1(tx,ty) + tx\frac{\partial F_1}{\partial x}(tx,ty) + ty\frac{\partial F_1}{\partial y}(tx,ty)dt \\ &= \int_0^1 \frac{d}{dt}\left[tF_1(tx,ty)\right]dt = \left[tF_1(tx,ty)\right]_{t=0}^{t=1} = F_1(x,y).\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{\partial \varphi}{\partial y}(x,y) =& \frac{\partial}{\partial y}\left[\int_0^1 xF_1(tx,ty) + yF_2(tx,ty)dt\right] \\ &= \int_0^1 \frac{\partial}{\partial y}\left[xF_1(tx,ty) + yF_2(tx,ty)\right]dt \\ &= \int_0^1 x\left(\frac{\partial F_1}{\partial x}(tx,ty)\frac{\partial}{\partial y}\left[tx\right] + \frac{\partial F_1}{\partial y}(tx,ty)\frac{\partial}{\partial y}\left[ty\right]\right) \\ &\qquad + F_2(tx,ty) + y\left(\frac{\partial F_2}{\partial x}(tx,ty)\frac{\partial}{\partial y}\left[tx\right] + \frac{\partial F_2}{\partial y}(tx,ty)\frac{\partial}{\partial y}\left[ty\right]\right)dt \\ &= \int_0^1 F_2(tx,ty) + x\frac{\partial F_1}{\partial y}(tx,ty)t + y\frac{\partial F_2}{\partial y}(ty,ty)tdt\end{aligned}$$

Using that $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ and the calculation above, we obtain

$$\begin{aligned}\frac{\partial \varphi}{\partial x}(x,y) &= \int_0^1 F_2(tx,ty) + tx\frac{\partial F_2}{\partial x}(tx,ty) + ty\frac{\partial F_2}{\partial y}(tx,ty)dt \\ &= \int_0^1 \frac{d}{dt}\left[tF_2(tx,ty)\right]dt = \left[tF_2(tx,ty)\right]_{t=0}^{t=1} = F_2(x,y).\end{aligned}$$

Thus we have shown that φ is indeed a potential of F.

We set

$$\varphi(x,y) = \int_0^1 x\cdot 2\cdot tx\cdot ty + y((tx)^2 + ty)dt = 3x^2y\int_0^1 t^2dt + y^2\int_0^1 tdt = x^2y + \frac{1}{2}y^2,$$

which, by the previous point, is a potential of F.

The formula from the course for an open set Ω star-shaped with respect to x_0 and a vector field $F\colon \Omega \to \mathbb{R}^n$ is

$$\varphi(x,y) = \int_0^1 \langle x - x_0, F(x_0 + t(x - x_0))\rangle\, dt.$$

This is the same formula but with $x_0 = 0$.