EPFL - Fall 2025 Dr. Pablo Antolin
Analysis III – MATH-202(c) – EL GM MX **Exercises**
Correction 3 September 23 2025

Exercise 1.

1. Γ is parameterized by $\gamma\colon [a,b] \to \mathbb{R}^2$ defined by $\gamma(t) = (t, f(t))$. Thus, $\gamma'(t) = (1, f'(t))$ and $|\gamma'(t)| = \sqrt{1^2 + (f'(t))^2}$. Finally,
$$\text{length}(\Gamma) = \int_a^b \sqrt{1 + (f'(t))^2} dt,$$
which is the desired result.
2. If $f(x) = \cosh(x)$, then, $f'(x) = \sinh(x)$. Moreover,
$$\begin{aligned}\sqrt{1 + (f'(t))^2} =& \sqrt{1 + \sinh(t)^2} = \sqrt{1 + \left(\frac{e^t - e^{-t}}{2}\right)^2} = \sqrt{1 + \frac{e^{2t} - 2 + e^{-2t}}{4}} = \sqrt{\frac{e^{2t} + 2 + e^{-2t}}{4}} \\ =& \sqrt{\left(\frac{e^t + e^{-t}}{2}\right)^2} = \sqrt{\cosh(t)^2} = |\cosh(t)| \overset{\cosh(t)\geq 0}{=} \cosh(t).\end{aligned}$$
and therefore,
$$\text{length}(\Gamma) = \int_0^1 \cosh(t) dt = [\sinh(t)]_0^1 = \sinh(1) - \sinh(0) = \sinh(1).$$
3. We parameterize Γ with $\gamma\colon [a,b] \to \mathbb{R}^2$ defined by $\gamma(t) = (r(t)\cos(t), r(t)\sin(t))$. Then,
$$\begin{aligned}\gamma'(t) =& (r'(t)\cos(t) - r(t)\sin(t), r'(t)\sin(t) + r(t)\cos(t)) \\ |\gamma'(t)|^2 =& (r'(t)\cos(t) - r(t)\sin(t))^2 + (r'(t)\sin(t) + r(t)\cos(t))^2 \\ =& r'(t)^2\cos^2(t) - 2r'(t)r(t)\cos(t)\sin(t) + r(t)^2\sin^2(t) \\ &+ r'(t)^2\sin^2(t) + 2r'(t)r(t)\cos(t)\sin(t) + r(t)^2\cos^2(t) \\ =& r'(t)^2 + r(t)^2\end{aligned}$$
and therefore,
$$\text{length}(\Gamma) = \int_a^b \sqrt{r'(t)^2 + r(t)^2} dt$$

Exercise 2.
Since f is a scalar field, the definition of the line integral is
$$\int_\Gamma f dl = \int_a^b f(\gamma(t))|\gamma'(t)| dt.$$

We have

$$
\begin{aligned}
f(\gamma(t)) =& \cos^2(t) + \sin^2(t) + \sqrt{2\frac{1}{2}t^2} = 1 + |t| \overset{t\in[0,1]}{=} 1 + t \\
\gamma'(t) =& (-\sin(t), \cos(t), t) \\
|\gamma'(t)| =& \sqrt{\sin^2(t) + \cos^2(t) + t^2} = \sqrt{1+t^2}
\end{aligned}
$$

Variant 1 : Calculation with the hint. Thus,

$$
\begin{aligned}
\int_\Gamma f dl =& \int_0^1 (1+t)\sqrt{1+t^2}dt = \int_0^1 \sqrt{1+t^2}dt + \int_0^1 t\sqrt{1+t^2}dt \\
=& \left[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log\left(t+\sqrt{1+t^2}\right)\right]_0^1 + \left[\frac{1}{3}\left(1+t^2\right)^{\frac{3}{2}}\right]_0^1 \\
=& \frac{1}{2}\sqrt{2} + \frac{1}{2}\log\left(1+\sqrt{2}\right) - 0\cdot\sqrt{1} - \frac{1}{2}\log\left(0+\sqrt{1}\right) + \frac{1}{3}2^{\frac{3}{2}} - \frac{1}{3}\cdot 1 \\
=& \frac{7}{6}\sqrt{2} - \frac{1}{3} + \frac{1}{2}\log\left(1+\sqrt{2}\right)
\end{aligned}
$$

Variant 2 : Change of variables.

Using that $\cosh^2(t) - \sinh^2(t) = 1$, we have

$$
\begin{aligned}
\int_0^1 \sqrt{1+t^2}dt \overset{t=\sinh(u)}{=}& \int_0^{\operatorname{arcsinh}(1)} \sqrt{1+\sinh^2(u)}\cosh(u)du = \int_0^{\operatorname{arcsinh}(1)} \sqrt{\cosh^2(u)}\cosh(u)du \\
\overset{\cosh(u)\geq 0}{=}& \int_0^{\operatorname{arcsinh}(1)} \cosh^2(u)du = \int_0^{\operatorname{arcsinh}(1)} \left(\frac{e^u+e^{-u}}{2}\right)^2 du \\
=& \frac{1}{4}\int_0^{\operatorname{arcsinh}1} e^{2u} + 2 + e^{-2u}du = \frac{1}{4}\left[\frac{1}{2}e^{2u} + 2u - \frac{1}{2}e^{-2u}\right]_0^{\operatorname{arcsinh}(1)} \\
=& \frac{1}{8}e^{2\operatorname{arcsinh}(1)} - \frac{1}{8}e^{-2\operatorname{arcsinh}(1)} + \frac{1}{2}\operatorname{arcsinh}(1)
\end{aligned}
$$

Let us still find the value of $\operatorname{arcsinh}(1)$.

If $a = \operatorname{arcsinh}(b)$, we have

$$
\begin{aligned}
& \sinh(a) = b \\
\Leftrightarrow\ & \frac{e^a - e^{-a}}{2} = b \\
\Leftrightarrow\ & (e^a)^2 - 2be^a - 1 = 0 \\
\Rightarrow\ & e^a = \frac{2b \pm \sqrt{4b^2+4}}{2} = b \pm \sqrt{1+b^2}
\end{aligned}
$$

since $e^a \geq 0$ and $b - \sqrt{1+b^2} \leq 0$, we must have $e^a = b + \sqrt{1+b^2}$ and therefore $a = \log\left(b + \sqrt{1+b^2}\right)$. Thus,

$$
\operatorname{arcsinh}(b) = \log\left(b + \sqrt{1+b^2}\right) \qquad \Rightarrow \qquad \operatorname{arcsinh}(1) = \log\left(1+\sqrt{2}\right)
$$

We therefore conclude

$$\begin{aligned}\int_0^1 \sqrt{1+t^2}dt =&\frac{1}{8}e^{2\log\left(1+\sqrt{2}\right)} - \frac{1}{8}e^{-2\log\left(1+\sqrt{2}\right)} + \frac{1}{2}\log\left(1+\sqrt{2}\right)\\ =&\frac{1}{8}\left(1+\sqrt{2}\right)^2 - \frac{1}{8\left(1+\sqrt{2}\right)^2} + \frac{1}{2}\log\left(1+\sqrt{2}\right)\\ =&\frac{3+2\sqrt{2}}{8} - \frac{1}{8(3+2\sqrt{2})} + \frac{1}{2}\log\left(1+\sqrt{2}\right)\\ =&\frac{3}{8} + \frac{\sqrt{2}}{4} - \frac{3-2\sqrt{2}}{8(9-8)} + \frac{1}{2}\log\left(1+\sqrt{2}\right)\\ =&\frac{\sqrt{2}}{2} + \frac{1}{2}\log\left(1+\sqrt{2}\right)\end{aligned}$$

Finally,

$$\begin{aligned}\int_\Gamma f dl = \int_0^1 (1+t)\sqrt{1+t^2}dt = \int_0^1 \sqrt{1+t^2}dt + \int_0^1 t\sqrt{1+t^2}dt\\ =&\frac{\sqrt{2}}{2} + \frac{1}{2}\log\left(1+\sqrt{2}\right) + \left[\frac{1}{3}\left(1+t^2\right)^{\frac{3}{2}}\right]_0^1\\ =&\frac{\sqrt{2}}{2} + \frac{1}{2}\log\left(1+\sqrt{2}\right) + + \frac{1}{3}2^{\frac{3}{2}} - \frac{1}{3}\cdot 1\\ =&\frac{7}{6}\sqrt{2} - \frac{1}{3} + \frac{1}{2}\log\left(1+\sqrt{2}\right)\end{aligned}$$

Exercise 3.

Since F is a vector field, the definition of the line integral is

$$\int_\Gamma F \cdot dl = \int_a^b \langle F(\gamma(t)), \gamma'(t)\rangle \, dt.$$

Moreover, for vector fields, the sign of the line integral depends on the direction of traversal. We perform the calculations for the directions of traversal given by the parameterizations that are given in the definitions of the curves Γ_i, but it would also be correct to parameterize differently and obtain the same thing but with the opposite sign.

Let us start with Γ_1 :

We parameterize Γ_1 with $\gamma_1 \colon [0,1] \to \mathbb{R}^2$ defined by $\gamma_1(t) = (t,t)$.

We then have,

$$\begin{aligned}F(\gamma_1(t)) =&(t^2, t^2 - t)\\ \gamma_1'(t) =&(1,1)\\ \int_{\Gamma_1} F\cdot dl =& \int_0^1 \left\langle (t^2, t^2-t),(1,1)\right\rangle dt = \int_0^1 2t^2 - tdt = \left[\frac{2}{3}t^3 - \frac{1}{2}t^2\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.\end{aligned}$$

Let us move to Γ_2 :

We parameterize Γ_2 with $\gamma_2 \colon [0,1] \to \mathbb{R}^2$ defined by $\gamma_2(t) = (t, e^t)$.

We then have,

$$\begin{aligned}F(\gamma_2(t)) =&(te^t, e^{2t} - t)\\ \gamma_2'(t) =&(1, e^t)\\ \int_{\Gamma_2} F\cdot dl =& \int_0^1 \left\langle (te^t, e^{2t}-t),(1,e^t)\right\rangle dt = \int_0^1 e^{3t}dt = \left[\frac{1}{3}e^{3t}\right]_0^1 = \frac{1}{3}\left(e^3-1\right)\end{aligned}$$

Let us finish with Γ_3 :

We parameterize Γ_3 with $\gamma_3 \colon [1,2] \to \mathbb{R}^2$ defined by $\gamma_2(t) = \left(\sqrt{t}, t^2\right)$

We then have,

$$\begin{aligned}F(\gamma_3(t)) =& \left(t^{\frac{5}{2}}, t^4 - \sqrt{t}\right)\\ \gamma_3'(t) =& \left(\frac{1}{2\sqrt{t}}, 2t\right)\\ \int_{\Gamma_3} F\cdot dl =& \int_1^2 \left\langle \left(t^{\frac{5}{2}}, t^4-\sqrt{t}\right), \left(\frac{1}{2\sqrt{t}}, 2t\right)\right\rangle dt = \int_1^2 \frac{1}{2}t^2 + 2t^5 - 2t^{\frac{3}{2}}dt = \left[\frac{1}{6}t^3 + \frac{1}{3}t^6 - \frac{4}{5}t^{\frac{5}{2}}\right]_1^2\\ =&\frac{4}{3} + \frac{64}{3} - \frac{16}{5}\sqrt{2} - \frac{1}{6} - \frac{1}{3} + \frac{4}{5} = \frac{40+640-5-10+24}{30} - \frac{16}{5}\sqrt{2} = \frac{689}{30} - \frac{16}{5}\sqrt{2}\end{aligned}$$

Exercise 4.

1. We start by parameterizing Γ. We observe the term $x^2+y^2=1$ and switch to cylindrical coordinates: $(x,y,z) = (r\cos\theta, r\sin\theta, z)$, $r \geq 0$, $\theta \in [0, 2\pi]$, $z \in \mathbb{R}$. Note that any interval of length 2π works for θ. We chose $[0,2\pi]$, but we could have chosen $[-\pi, \pi]$, or something else.

   Our conditions become

$$\begin{aligned}x^2+y^2=1 \Leftrightarrow \quad & r = 1\\ z = 0 \quad & \text{does not change}\end{aligned}$$

r and z are fixed, leaving only θ as the variable with no conditions on it. We therefore parameterize Γ with $\gamma\colon [0,2\pi] \to \mathbb{R}^3$ defined by

$$\gamma(t) = (\cos t, \sin t, 0).$$

We then have,

$$\begin{aligned} F(\gamma(t)) &= (\cos t, 0, \sin t) \\ \gamma'(t) &= (-\sin t, \cos t, 0) \\ \int_\Gamma F \cdot dl &= \int_0^{2\pi} \langle (\cos t, 0, \sin t), (-\sin t, \cos t, 0)\rangle\, dt = \int_0^{2\pi} -\cos t \sin t dt = \left[\frac{1}{2}\cos^2(t)\right]_0^{2\pi} = 0. \end{aligned}$$

Alternatively, we can find an antiderivative of $\cos t \sin t$ using Euler's formulas:

$$\cos t \sin t = \frac{e^{it}+e^{-it}}{2}\frac{e^{it}-e^{-it}}{2i} = \frac{1}{2}\frac{e^{2it}-e^{-2it}}{2i} = \frac{1}{2}\sin(2t)$$

whose antiderivative is known.

2. <u>*Variant 1*</u> : x is the variable of the parameterization.

This is the most natural since y and z can be written as functions of x and we have an interval for x.

We therefore parameterize Γ with $\gamma\colon [0,1] \to \mathbb{R}^3$ defined by

$$\gamma(t) = \left(t, e^t, t\right)$$

We then have,

$$\begin{aligned} F(\gamma(t)) &= \left(t, e^t, t\right) \\ \gamma'(t) &= \left(1, e^t, 1\right) \\ \int_\Gamma F \cdot dl &= \int_0^1 \left\langle \left(t, e^t, t\right), \left(1, e^t, 1\right)\right\rangle dt = \int_0^1 e^{2t} + 2t dt = \left[\frac{1}{2}e^{2t} + t^2\right]_0^1 = \frac{1}{2}e^2 + 1 - \frac{1}{2} = \frac{e^2+1}{2} \end{aligned}$$

<u>*Variant 2*</u> : y is the variable of the parameterization.

Our conditions are equivalent to

$$x = \log(y) \qquad z = x = \log(y).$$

Moreover, for $x \in [0,1]$, we need $y \in [1,e]$. Thus, we can parameterize Γ with $\gamma\colon [1,e] \to \mathbb{R}^3$ defined by

$$\gamma(t) = (\log(t), t, \log(t)).$$

We then have,

$$\begin{aligned}F(\gamma(t)) &= (\log(t), t, \log(t))\\ \gamma'(t) &= \left(\frac{1}{t}, 1, \frac{1}{t}\right)\\ \int_\Gamma F\cdot dl &= \int_1^e \left\langle (\log(t), t, \log(t))\,, \left(\frac{1}{t}, 1, \frac{1}{t}\right)\right\rangle dt = \int_1^e 2\frac{\log(t)}{t} + tdt = \left[\log(t)^2 + \frac{1}{2}t^2\right]_1^e\\ &= 1 + \frac{1}{2}e^2 - \frac{1}{2} = \frac{1+e^2}{2}\end{aligned}$$

Variant 3 : z is the variable of the parameterization.

We then obtain exactly the same result as in variant 1.

Exercise 5.
1- We give here the rigorous proof that γ is a parameterization of Γ. We must show three things:

- γ is injective on $]0,\pi[$
- $\forall t \in [0,\pi]$, $\gamma(t) \in \Gamma$
- $\forall (x,y) \in \Gamma$, $\exists t \in [0,\pi]$ such that $(x,y) = \gamma(t)$.

This level of detail is not necessary in the exam: for parameterizations, you will generally be asked to find one, not to prove that it is indeed one. Nevertheless, we do it here once to see what it looks like.

1. Let $t, s \in]0,\pi[$ such that

$$\begin{cases} \sin(t) = \sin(s) & (1)\\ \sin(2t) = \sin(2s) & (2)\end{cases}$$

Then, (1) $\Rightarrow s = \pi - t$ or $s = t$. if $s = t$, we have the desired result. If $t = \pi - s$, plugging this into (2), we get $\sin(2\pi - 2s) = \sin(2s)$. Since $\sin(2\pi - 2s) = -\sin(2s)$, we deduce that $\sin(2s) = 0$. Now, $2s \in]0, 2\pi[$ and the only solution of $\sin(x) = 0$ in $]0, 2\pi[$ is $x = \pi$. Thus, $2s = \pi$ and therefore $s = \frac{\pi}{2}$ and thus $t = \pi - s = \pi - \frac{\pi}{2} = \frac{\pi}{2} = s$, which is the desired result.

2. We have for all t,

$$\begin{aligned}\sin^2(2t)+4\cos^4(t)-4\cos^2(t) &= \left(\frac{e^{i2t}-e^{-i2t}}{2i}\right)^2+4\left(\frac{e^{it}+e^{-it}}{2}\right)^4-4\left(\frac{e^{it}+e^{-it}}{2}\right)^2\\ &=\frac{-1}{4}\left(e^{i4t}-2+e^{-i4t}\right)+\frac{1}{4}\left(e^{i4t}+4e^{i3t}e^{-it}+\binom{4}{2}e^{i2t}e^{-i2t}+4e^{it}e^{-i3t}+e^{-i4t}\right)\\ &\quad-\left(e^{i2t}+2+e^{-i2t}\right)\\ &=\frac{1}{2}+\frac{6}{4}-2=0\end{aligned}$$

3. Let $(x,y)\in\Gamma$. Let us start by noting that

$$\begin{aligned}y^2+4x^4-4x^2=0 &\Longrightarrow 4x^4-4x^2=-y^2\\ &\Longrightarrow 4x^4-4x^2\leq 0\\ &\Longrightarrow 4x^2(x^2-1)\leq 0\\ &\overset{x\geq 0}{\Longrightarrow} x\in[0,1].\end{aligned}$$

Moreover, we have

$$y^2=4x^2-4x^4=4x^2(1-x^2)\overset{x\in[0,1]}{\Longrightarrow}|y|=2x\sqrt{1-x^2}$$

We now distinguish 2 cases based on the sign of y. The idea of the distinction comes either from the drawing of the curve where we see that on the first half of the path $y\geq 0$ and on the second half $y\leq 0$, or by noting that for $t\in[0,\pi]$, $\sin(2t)\geq 0\Leftrightarrow 2t\in[0,\pi]\Leftrightarrow t\in[0,\pi/2]$.

Case 1 : $y\geq 0$.

Since $x\in[0,1]$, there exists $t\in[0,\pi/2]$ such that $x=\sin(t)$. Moreover, we then have, since $y\geq 0$,

$$\begin{aligned}y=&2x\sqrt{1-x^2}=2\sin(t)\sqrt{1-\sin^2(t)}=2\sin(t)\sqrt{\cos^2(t)}\overset{t\in[0,\pi/2]\Rightarrow\cos(t)\geq 0}{=}2\sin(t)\cos(t)\\ =&2\frac{e^{it}-e^{-it}}{2i}\frac{e^{it}+e^{-it}}{2}=\frac{e^{i2t}-e^{-i2t}}{2i}=\sin(2t),\end{aligned}$$

and thus $(x,y)=\gamma(t)$.

Case 2 : $y\leq 0$.

Since $x\in[0,1]$, there exists $t\in[\pi/2,\pi]$ such that $x=\sin(t)$. Moreover, we then have, since $y\leq 0$,

$$\begin{aligned}y=&-2x\sqrt{1-x^2}=-2\sin(t)\sqrt{1-\sin^2(t)}\\ =&-2\sin(t)\sqrt{\cos^2(t)}\overset{t\in[\pi/2,\pi]\Rightarrow\cos(t)\leq 0}{=}-2\sin(t)(-\cos(t))=\sin(2t),\end{aligned}$$

and thus $(x,y)=\gamma(t)$.

This shows that γ is indeed a parameterization of Γ.

2- The curve is given by $y = \pm 2x\sqrt{1-x^2}$. One can choose

$$\Gamma = \{\gamma(t) = (\sin t, \sin(2t)), t \in [0, \pi]\}.$$

(Note that the orientation of Γ is not explicitly given in the problem; we have chosen the counterclockwise orientation). We therefore find

$$\begin{aligned}\int_\Gamma F \cdot dl &= \int_0^\pi (\sin t + \sin(2t), -\sin t) \cdot (\cos t, 2\cos(2t))dt \\ &= \int_0^\pi \{-2\cos(2t)\sin t + \cos t[\sin t + \sin(2t)]\}dt = \frac{8}{3}\end{aligned}$$

Exercise 6.
Let $\gamma : [a, b] \to \Gamma$ be a parameterization of Γ, we have $\gamma(a) = A$, $\gamma(b) = B$. Moreover $F(\gamma(t)) = m\gamma''(t)$. We thus find

$$\begin{aligned}\int_\Gamma F \cdot dl &= \int_a^b m\gamma''(t) \cdot \gamma'(t)dt = m \int_a^b \frac{d}{dt}\left[\frac{1}{2}\left\|\gamma'(t)\right\|^2\right] dt \\ &= \frac{m}{2}\left\|\gamma'(b)\right\|^2 - \frac{m}{2}\left\|\gamma'(a)\right\|^2,\end{aligned}$$

which is nothing other than the kinetic energy variation.