

Exercise 1.

1. Let's start with F_1 . We have

$$\begin{aligned}\frac{\partial F_1}{\partial x}(x, y, z) &= y^2 z \cos(xz) \\ \frac{\partial F_1}{\partial y}(x, y, z) &= 2y \sin(xz) \\ \frac{\partial F_1}{\partial z}(x, y, z) &= xy^2 \cos(xz)\end{aligned}$$

and therefore

$$\nabla F_1(x, y, z) = (y^2 z \cos(xz), 2y \sin(xz), xy^2 \cos(xz))$$

Let's move on to F_2 . We have

$$\begin{aligned}\frac{\partial F_2}{\partial x}(x, y, z) &= -2xe^y \sin(x^2 + z) \\ \frac{\partial F_2}{\partial y}(x, y, z) &= e^y \cos(x^2 + z) \\ \frac{\partial F_2}{\partial z}(x, y, z) &= -e^y \sin(x^2 + z)\end{aligned}$$

and therefore

$$\nabla F_2(x, y, z) = (-2xe^y \sin(x^2 + z), e^y \cos(x^2 + z), -e^y \sin(x^2 + z))$$

Let's finish with F_3 . We have

$$\begin{aligned}\frac{\partial F_3}{\partial x}(x, y, z) &= \frac{\frac{\partial}{\partial x} [2 + \cos(xy)]}{2 + \cos(xy)} = \frac{-y \sin(xy)}{2 + \cos(xy)} \\ \frac{\partial F_3}{\partial y}(x, y, z) &= \frac{\frac{\partial}{\partial y} [2 + \cos(xy)]}{2 + \cos(xy)} = \frac{-x \sin(xy)}{2 + \cos(xy)} \\ \frac{\partial F_3}{\partial z}(x, y, z) &= 0\end{aligned}$$

and therefore

$$\nabla F_3(x, y, z) = \left(\frac{-y \sin(xy)}{2 + \cos(xy)}, \frac{-x \sin(xy)}{2 + \cos(xy)}, 0 \right)$$

2. We have

$$\begin{aligned}\operatorname{div} F &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= y^2 z \cos(xz) + e^y \cos(x^2 + z) + 0\end{aligned}$$

3. We have

$$\begin{aligned}\operatorname{rot} F &= \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \\ &= \left(\frac{-x \sin(xy)}{2 + \cos(xy)} + e^y \sin(x^2 + z), xy^2 \cos(xz) - \frac{-y \sin(xy)}{2 + \cos(xy)}, -2xe^y \sin(x^2 + z) - 2y \sin(xz) \right)\end{aligned}$$

Exercise 2.

The goal is to use the following rules:

- ∇ applies to scalar fields¹
 - ∇ gives a vector field
 - rot and div apply to vector fields.
 - rot gives a vector field (since we are in \mathbb{R}^3 . If we were in \mathbb{R}^2 it would be a scalar field.)
 - div gives a scalar field.
1. *OK*: ∇ is defined for scalar fields. We obtain a vector field.
 2. *OK*: ∇f is defined and is a vector field. We can then multiply the vector field ∇f by the scalar field f : we use pointwise scalar multiplication of a vector.
 3. *OK*: F and ∇f are vector fields, so we can take their dot product.
 4. *Oh hell nah!*: The divergence only applies to vector fields; f is a scalar field.
 5. *OK*: $f \cdot F$, obtained by scalar multiplication of the vector field F by f , is a vector field. The divergence indeed applies to vector fields.
 6. *OK*: $f \cdot F$, obtained by scalar multiplication of the vector field F by f , is a vector field. The curl indeed applies to vector fields.

¹Here we interpret ∇ as the *gradient*. In reality, ∇ also applies to functions $g: \mathbb{R}^m \rightarrow \mathbb{R}^n$ and is interpreted as the *Jacobian matrix*.

7. *Oh hell nah!*: The curl only applies to vector fields; f is a scalar field.
8. *OK*: F is a vector field and, since we are in \mathbb{R}^3 , $\text{rot } F$ is also a vector field. Then, we multiply by the scalar f .
9. *Oh hell nah!*: $\text{div } F$ is a scalar field and the curl requires a vector field.

Exercise 3.

We have that f is the composition of several functions. Therefore, by proceeding step by step,

$$\begin{aligned} \frac{\partial f}{\partial x_j}(x) &= \frac{\partial}{\partial x_j} [r(x)^{-1}] = -r(x)^{-2} \frac{\partial r}{\partial x_j}(x) = \frac{-1}{\sum_{i=1}^n (x_i - a_i)^2} \frac{\partial}{\partial x_j} \left[\sqrt{\sum_{i=1}^n (x_i - a_i)^2} \right] \\ &= \frac{-1}{\sum_{i=1}^n (x_i - a_i)^2} \frac{1}{2} \frac{1}{\sqrt{\sum_{i=1}^n (x_i - a_i)^2}} \frac{\partial}{\partial x_j} \left[\sum_{i=1}^n (x_i - a_i)^2 \right] \\ &= \frac{-1}{2 \left(\sum_{i=1}^n (x_i - a_i)^2 \right)^{\frac{3}{2}}} \sum_{i=1}^n \frac{\partial}{\partial x_j} [(x_i - a_i)^2] \end{aligned}$$

Note at this stage that if $i \neq j$, then $(x_i - a_i)^2$ does not depend on x_j , and therefore $\frac{\partial}{\partial x_j} [(x_i - a_i)^2] = 0$, whereas if $i = j$, $\frac{\partial}{\partial x_j} [(x_i - a_i)^2] = 2(x_j - a_j)$.

Thus,

$$\frac{\partial f}{\partial x_j}(x) = \frac{-1}{2 \left(\sum_{i=1}^n (x_i - a_i)^2 \right)^{\frac{3}{2}}} 2(x_j - a_j) = \frac{a_j - x_j}{r(x)^3}$$

We move on to the second-order derivative,

$$\begin{aligned} \frac{\partial^2 f}{\partial x_j^2}(x) &= \frac{\partial}{\partial x_j} \left[\frac{a_j - x_j}{r(x)^3} \right] = -\frac{1}{r(x)^3} + (a_j - x_j) \frac{\partial}{\partial x_j} [r(x)^{-3}] = -\frac{1}{r(x)^3} + (a_j - x_j)(-3)r(x)^{-4} \frac{\partial r}{\partial x_j}(x) \\ &= -\frac{1}{r(x)^3} + (a_j - x_j)(-3)r(x)^{-4} \frac{\partial}{\partial x_j} \left[\sqrt{\sum_{i=1}^n (x_i - a_i)^2} \right] \\ &= -\frac{1}{r(x)^3} + \frac{3(x_j - a_j)}{r(x)^4} \frac{1}{2} \frac{1}{\sqrt{\sum_{i=1}^n (x_i - a_i)^2}} 2(x_j - a_j) \\ &= -\frac{1}{r(x)^3} + 3 \frac{(x_j - a_j)^2}{r(x)^5}. \end{aligned}$$

Finally,

$$\begin{aligned}
\Delta f(x) &= \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}(x) \\
&= \sum_{j=1}^n \left(-\frac{1}{r(x)^3} + 3 \frac{(x_j - a_j)^2}{r(x)^5} \right) \\
&= -\frac{n}{r(x)^3} + \frac{3}{r(x)^5} \underbrace{\sum_{j=1}^n (x_j - a_j)^2}_{=r(x)^2} \\
&= \frac{3-n}{r(x)^3}
\end{aligned}$$

Exercise 4.

For the first three points, we carry out the demonstration in the more general case where $\Omega \subset \mathbb{R}^n$ with arbitrary n . For the last two, they are only valid for $n = 3$.

1. We have

$$\begin{aligned}
\operatorname{div}(f\nabla g) &= \sum_{j=1}^n \frac{\partial}{\partial x_j} [(f\nabla g)_j] = \sum_{j=1}^n \frac{\partial}{\partial x_j} \left[f \frac{\partial g}{\partial x_j} \right] = \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_j} + f \frac{\partial^2 g}{\partial x_j^2} \right) \\
&= \underbrace{\sum_{j=1}^n \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_j}}_{=\langle \nabla f, \nabla g \rangle} + f \underbrace{\sum_{j=1}^n \frac{\partial^2 g}{\partial x_j^2}}_{=\Delta g},
\end{aligned}$$

which is the desired result.

2. Since

$$\frac{\partial}{\partial x_j} [fg] = f \frac{\partial g}{\partial x_j} + g \frac{\partial f}{\partial x_j},$$

we have

$$\begin{aligned}
\nabla(fg) &= \left(f \frac{\partial g}{\partial x_1} + g \frac{\partial f}{\partial x_1}, \dots, f \frac{\partial g}{\partial x_n} + g \frac{\partial f}{\partial x_n} \right) \\
&= f \left(\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n} \right) + g \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) \\
&= f\nabla g + g\nabla f.
\end{aligned}$$

3. We have

$$\begin{aligned}\operatorname{div}(fF) &= \sum_{j=1}^n \frac{\partial}{\partial x_j} [(fF)_j] = \sum_{j=1}^n \frac{\partial}{\partial x_j} [fF_j] = \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} F_j + f \frac{\partial F_j}{\partial x_j} \right) \\ &= \underbrace{\sum_{j=1}^n \frac{\partial f}{\partial x_j} F_j}_{=\langle \nabla f, F \rangle} + f \underbrace{\sum_{j=1}^n \frac{\partial F_j}{\partial x_j}}_{=\operatorname{div} F},\end{aligned}$$

which is the desired result.

4. We have

$$\operatorname{rot} F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).$$

Thus, (note that since F is C^2 , when we take second-order partial derivatives, the order of the variables with respect to which we differentiate does not matter)

$$\begin{aligned}\operatorname{rot} \operatorname{rot} F &= \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix} \\ &= \left(\frac{\partial^2 F_2}{\partial x \partial y} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial^2 z} + \frac{\partial^2 F_3}{\partial x \partial z} \right) e_1 \\ &\quad + \left(-\frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial y \partial z} - \frac{\partial^2 F_2}{\partial z^2} \right) e_2 \\ &\quad + \left(\frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y \partial z} \right) e_3.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{\partial}{\partial x} [\operatorname{div} F] &= \frac{\partial}{\partial x} \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right] \\ &= \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} \\ \frac{\partial}{\partial y} [\operatorname{div} F] &= \frac{\partial}{\partial y} \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right] \\ &= \frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y \partial z} \\ \frac{\partial}{\partial z} [\operatorname{div} F] &= \frac{\partial}{\partial z} \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right] \\ &= \frac{\partial^2 F_1}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} + \frac{\partial^2 F_3}{\partial z^2}\end{aligned}$$

Thus,

$$\nabla \operatorname{div} F = \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z}, \frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y \partial z}, \frac{\partial^2 F_1}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} + \frac{\partial^2 F_3}{\partial z^2} \right)$$

and therefore

$$\begin{aligned} \nabla \operatorname{div} F - \operatorname{rot} \operatorname{rot} F &= \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}, \frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2}, \frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2} \right) \\ &= \Delta F. \end{aligned}$$

By rearranging the terms, we obtain the desired result.

5. We have

$$\begin{aligned} \operatorname{rot}(fF) &= \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}[fF_3] - \frac{\partial}{\partial z}[fF_2], \frac{\partial}{\partial z}[fF_1] - \frac{\partial}{\partial x}[fF_3], \frac{\partial}{\partial x}[fF_2] - \frac{\partial}{\partial y}[fF_1] \right) \\ &= f \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \\ &\quad + \left(\frac{\partial f}{\partial y} F_3 - \frac{\partial f}{\partial z} F_2, \frac{\partial f}{\partial z} F_1 - \frac{\partial f}{\partial x} F_3, \frac{\partial f}{\partial x} F_2 - \frac{\partial f}{\partial y} F_1 \right) \\ &= r \operatorname{rot} F + \left(\frac{\partial f}{\partial y} F_3 - \frac{\partial f}{\partial z} F_2, \frac{\partial f}{\partial z} F_1 - \frac{\partial f}{\partial x} F_3, \frac{\partial f}{\partial x} F_2 - \frac{\partial f}{\partial y} F_1 \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \nabla f \wedge F &= \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial f}{\partial y} F_3 - \frac{\partial f}{\partial z} F_2, \frac{\partial f}{\partial z} F_1 - \frac{\partial f}{\partial x} F_3, \frac{\partial f}{\partial x} F_2 - \frac{\partial f}{\partial y} F_1 \right). \end{aligned}$$

By comparing with what we obtained in the calculation of $\operatorname{rot}(fF)$, we indeed get

$$\operatorname{rot}(fF) = \nabla f \wedge F + f \operatorname{rot} F,$$

which is the desired result.

Exercise 5. 1. Version 1 : Direct calculation.

We have

$$\begin{aligned}
\frac{\partial g}{\partial r}(r, \theta) &= \frac{\partial}{\partial r} [f(r \cos \theta, r \sin \theta)] \\
&= \frac{\partial f}{\partial x}(r \cos \theta, r \sin \theta) \frac{\partial}{\partial r} [r \cos \theta] + \frac{\partial f}{\partial y}(r \cos \theta, r \sin \theta) \frac{\partial}{\partial r} [r \sin \theta] \\
&= \frac{\partial f}{\partial x}(r \cos \theta, r \sin \theta) \cos \theta + \frac{\partial f}{\partial y}(r \cos \theta, r \sin \theta) \sin \theta \\
\frac{\partial^2 g}{\partial r^2}(r, \theta) &= \frac{\partial}{\partial r} \left[\frac{\partial f}{\partial x}(r \cos \theta, r \sin \theta) \cos \theta + \frac{\partial f}{\partial y}(r \cos \theta, r \sin \theta) \sin \theta \right] \\
&= \frac{\partial}{\partial r} \left[\frac{\partial f}{\partial x}(r \cos \theta, r \sin \theta) \right] \cos \theta + \frac{\partial}{\partial r} \left[\frac{\partial f}{\partial y}(r \cos \theta, r \sin \theta) \right] \sin \theta \\
&= \left(\frac{\partial^2 f}{\partial x^2}(r \cos \theta, r \sin \theta) \frac{\partial}{\partial r} [r \cos \theta] + \frac{\partial^2 f}{\partial x \partial y}(r \cos \theta, r \sin \theta) \frac{\partial}{\partial r} [r \sin \theta] \right) \cos \theta \\
&\quad + \left(\frac{\partial^2 f}{\partial x \partial y}(r \cos \theta, r \sin \theta) \frac{\partial}{\partial r} [r \cos \theta] + \frac{\partial^2 f}{\partial y^2}(r \cos \theta, r \sin \theta) \frac{\partial}{\partial r} [r \sin \theta] \right) \sin \theta \\
&= \frac{\partial^2 f}{\partial x^2}(r \cos \theta, r \sin \theta) \cos^2 \theta + 2 \frac{\partial^2 f}{\partial x \partial y}(r \cos \theta, r \sin \theta) \cos \theta \sin \theta \\
&\quad + \frac{\partial^2 f}{\partial y^2}(r \cos \theta, r \sin \theta) \sin^2 \theta \\
\frac{\partial g}{\partial \theta}(r, \theta) &= \frac{\partial}{\partial \theta} [f(r \cos \theta, r \sin \theta)] \\
&= \frac{\partial f}{\partial x}(r \cos \theta, r \sin \theta) \frac{\partial}{\partial \theta} [r \cos \theta] + \frac{\partial f}{\partial y}(r \cos \theta, r \sin \theta) \frac{\partial}{\partial \theta} [r \sin \theta] \\
&= r \left(-\frac{\partial f}{\partial x}(r \cos \theta, r \sin \theta) \sin \theta + \frac{\partial f}{\partial y}(r \cos \theta, r \sin \theta) \cos \theta \right) \\
\frac{\partial^2 g}{\partial \theta^2}(r, \theta) &= \frac{\partial}{\partial \theta} \left[r \left(-\frac{\partial f}{\partial x}(r \cos \theta, r \sin \theta) \sin \theta + \frac{\partial f}{\partial y}(r \cos \theta, r \sin \theta) \cos \theta \right) \right] \\
&= r \left(-\frac{\partial}{\partial \theta} \left[\frac{\partial f}{\partial x}(r \cos \theta, r \sin \theta) \right] \sin \theta - \frac{\partial f}{\partial x}(r \cos \theta, r \sin \theta) \cos \theta \right. \\
&\quad \left. + \frac{\partial}{\partial \theta} \left[\frac{\partial f}{\partial y}(r \cos \theta, r \sin \theta) \right] \cos \theta - \frac{\partial f}{\partial y}(r \cos \theta, r \sin \theta) \sin \theta \right) \\
&= -r \sin \theta \left(\frac{\partial^2 f}{\partial x^2}(r \cos \theta, r \sin \theta) \frac{\partial}{\partial \theta} [r \cos \theta] + \frac{\partial^2 f}{\partial x \partial y}(r \cos \theta, r \sin \theta) \frac{\partial}{\partial \theta} [r \sin \theta] \right) \\
&\quad - r \cos \theta \frac{\partial f}{\partial x}(r \cos \theta, r \sin \theta) \\
&\quad + r \cos \theta \left(\frac{\partial^2 f}{\partial x \partial y}(r \cos \theta, r \sin \theta) \frac{\partial}{\partial \theta} [r \cos \theta] + \frac{\partial^2 f}{\partial y^2}(r \cos \theta, r \sin \theta) \frac{\partial}{\partial \theta} [r \sin \theta] \right) \\
&\quad - r \sin \theta \frac{\partial f}{\partial y}(r \cos \theta, r \sin \theta) \\
&= -r^2 \sin^2 \theta \frac{\partial^2 f}{\partial x^2}(r \cos \theta, r \sin \theta) - 2r^2 \cos \theta \sin \theta \frac{\partial^2 f}{\partial x \partial y}(r \cos \theta, r \sin \theta) \\
&\quad + r^2 \cos^2 \theta \frac{\partial^2 f}{\partial y^2}(r \cos \theta, r \sin \theta) \\
&\quad - r \cos \theta \frac{\partial f}{\partial x}(r \cos \theta, r \sin \theta) - r \sin \theta \frac{\partial f}{\partial y}(r \cos \theta, r \sin \theta)
\end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial^2 g}{\partial r^2} + \frac{1}{r} \frac{\partial g}{\partial r} + \frac{1}{r^2} \frac{\partial^2 g}{\partial \theta^2} &= \underbrace{(\cos^2 \theta + \sin^2 \theta)}_{=1} \frac{\partial^2 f}{\partial x^2}(r \cos \theta, r \sin \theta) + \underbrace{(\cos^2 \theta + \sin^2 \theta)}_{=1} \frac{\partial^2 f}{\partial y^2}(r \cos \theta, r \sin \theta) \\ &= \Delta f(r \cos \theta, r \sin \theta) \end{aligned}$$

Variante 2 : With physicists' tricks.

If $x(r, \theta) = r \cos \theta$ and $y(r, \theta) = r \sin \theta$, we have

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos \theta = \frac{x}{r} \\ \frac{\partial x}{\partial \theta} &= -r \sin \theta = -y \\ \frac{\partial y}{\partial r} &= \sin \theta = \frac{y}{r} \\ \frac{\partial y}{\partial \theta} &= r \cos \theta = x \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial g}{\partial r} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} = \frac{1}{r} \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right) \\ \frac{\partial^2 g}{\partial r^2} &= -\frac{1}{r^2} \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right) \\ &\quad + \frac{1}{r} \left(\frac{\partial x}{\partial r} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial f}{\partial y} + x \left(\frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial y}{\partial r} \right) + y \left(\frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial r} \right) \right) \\ &= -\frac{1}{r^2} \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right) \\ &\quad + \frac{1}{r} \left(\frac{x}{r} \frac{\partial f}{\partial x} + \frac{y}{r} \frac{\partial f}{\partial y} + x \left(\frac{\partial^2 f}{\partial x^2} \frac{x}{r} + \frac{\partial^2 f}{\partial x \partial y} \frac{y}{r} \right) + y \left(\frac{\partial^2 f}{\partial x \partial y} \frac{x}{r} + \frac{\partial^2 f}{\partial y^2} \frac{y}{r} \right) \right) \\ &= \frac{1}{r^2} \left(x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} \right) \\ \frac{\partial g}{\partial \theta} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} = -y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y} \\ \frac{\partial^2 g}{\partial \theta^2} &= -\frac{\partial y}{\partial \theta} \frac{\partial f}{\partial x} - y \left(\frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial y}{\partial \theta} \right) + \frac{\partial x}{\partial \theta} \frac{\partial f}{\partial y} + x \left(\frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial \theta} \right) \\ &= -x \frac{\partial f}{\partial x} - y \frac{\partial f}{\partial y} - y \left(-y \frac{\partial^2 f}{\partial x^2} + x \frac{\partial^2 f}{\partial x \partial y} \right) + x \left(-y \frac{\partial^2 f}{\partial x \partial y} + x \frac{\partial^2 f}{\partial y^2} \right) \\ &= -x \frac{\partial f}{\partial x} - y \frac{\partial f}{\partial y} + y^2 \frac{\partial^2 f}{\partial x^2} - 2xy \frac{\partial^2 f}{\partial x \partial y} + x^2 \frac{\partial^2 f}{\partial y^2} \end{aligned}$$

And therefore we conclude

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r} \frac{\partial g}{\partial r} + \frac{1}{r^2} \frac{\partial^2 g}{\partial \theta^2} = \underbrace{\frac{x^2 + y^2}{r^2}}_{=1} \frac{\partial^2 f}{\partial x^2} + \underbrace{\frac{x^2 + y^2}{r^2}}_{=1} \frac{\partial^2 f}{\partial y^2} = \Delta f$$

2. Let us define

$$g(r, \theta) = f(r \cos \theta, r \sin \theta) = r + \left(\arctan \frac{\sin \theta}{\cos \theta} \right)^2 = r + (\arctan \tan)^2 = r + \theta^2$$

Then,

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r} \frac{\partial g}{\partial r} + \frac{1}{r^2} \frac{\partial^2 g}{\partial \theta^2} = 0 + \frac{1}{r} \cdot 1 + \frac{1}{r^2} \cdot 2 = \frac{2+r}{r^2} = \Delta f(r \cos \theta, r \sin \theta).$$

Thus,

$$\Delta f(x, y) = \frac{2 + \sqrt{x^2 + y^2}}{x^2 + y^2}$$