EPFL - Fall 2025	Dr. Pablo Antolin
Analysis III – MATH-202(c) – EL GM MX	**Exercises**
Correction 1	September 9 2025

Exercise 1.

1. We have:

$$\begin{aligned}\frac{\partial}{\partial x_i}[\langle \boldsymbol{F}, \boldsymbol{G}\rangle] &= \frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n} F_j G_j\right] \\ &= \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left[F_j G_j\right] \\ &= \sum_{j=1}^{n} \left(\frac{\partial F_j}{\partial x_i} G_j + F_j \frac{\partial G_j}{\partial x_i}\right) \\ &= \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_i} G_j + \sum_{j=1}^{n} F_j \frac{\partial G_j}{\partial x_i},\end{aligned}$$

which leads to:

$$\frac{\partial}{\partial x_i}[\langle \boldsymbol{F}, \boldsymbol{G}\rangle] = \left\langle \frac{\partial \boldsymbol{F}}{\partial x_i}, \boldsymbol{G}\right\rangle + \left\langle \boldsymbol{F}, \frac{\partial \boldsymbol{G}}{\partial x_i}\right\rangle$$

2. Let us consider the first component. We use the subscript $(\cdot)_i$ to denote the i-th component of a given vector. We have:

$$\begin{aligned}\left(\frac{\partial}{\partial x_i}[\boldsymbol{F} \wedge \boldsymbol{G}]\right)_1 &= \frac{\partial}{\partial x_i}\left[(\boldsymbol{F} \wedge \boldsymbol{G})_1\right] \\ &= \frac{\partial}{\partial x_i}\left[F_2 G_3 - F_3 G_2\right] \\ &= \frac{\partial}{\partial x_i}\left[F_2 G_3\right] - \frac{\partial}{\partial x_i}\left[F_3 G_2\right] \\ &= \frac{\partial F_2}{\partial x_i} G_3 + F_2 \frac{\partial G_3}{\partial x_i} - \left(\frac{\partial F_3}{\partial x_i} G_2 + F_3 \frac{\partial G_2}{\partial x_i}\right) \\ &= \frac{\partial F_2}{\partial x_i} G_3 - \frac{\partial F_3}{\partial x_i} G_2 + F_2 \frac{\partial G_3}{\partial x_i} - F_3 \frac{\partial G_2}{\partial x_i} \\ &= \left(\frac{\partial \boldsymbol{F}}{\partial x_i}\right)_2 G_3 - \left(\frac{\partial \boldsymbol{F}}{\partial x_i}\right)_3 G_2 + F_2 \left(\frac{\partial \boldsymbol{G}}{\partial x_i}\right)_3 - F_3 \left(\frac{\partial \boldsymbol{G}}{\partial x_i}\right)_2 \\ &= \left(\frac{\partial \boldsymbol{F}}{\partial x_i} \wedge \boldsymbol{G}\right)_1 + \left(\boldsymbol{F} \wedge \frac{\partial \boldsymbol{G}}{\partial x_i}\right)_1 \\ &= \left(\frac{\partial \boldsymbol{F}}{\partial x_i} \wedge \boldsymbol{G} + \boldsymbol{F} \wedge \frac{\partial \boldsymbol{G}}{\partial x_i}\right)_1\end{aligned}$$

The exact same procedure holds true for the second and third components. Therefore, we show that

$$\frac{\partial}{\partial x_i}[\boldsymbol{F} \wedge \boldsymbol{G}] = \frac{\partial \boldsymbol{F}}{\partial x_i} \wedge \boldsymbol{G} + \boldsymbol{F} \wedge \frac{\partial \boldsymbol{G}}{\partial x_i}.$$

Exercise 2.

1. Polar coordinates:

$$\operatorname{Jac}(u)(r,\theta) = \det \nabla u(r,\theta) = \det \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} = r.$$

2. Spherical coordinates:

$$\begin{aligned} \operatorname{Jac}(u)(r,\theta,\varphi) &= \det \nabla u(r,\theta,\varphi) \\ &= \det \begin{pmatrix} \cos(\theta)\sin(\varphi) & -r\sin(\theta)\sin(\varphi) & r\cos(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) & r\cos(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi) \\ \cos(\varphi) & 0 & -r\sin(\varphi) \end{pmatrix} \\ &= -r^2\sin(\varphi). \end{aligned}$$

3. Cylindrical coordinates:

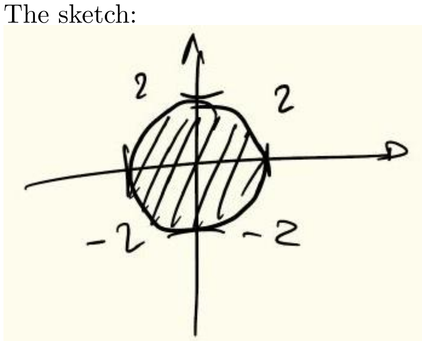
$$\operatorname{Jac}(u)(r,\theta,z) = \det \nabla u(r,\theta,z) = \det \begin{pmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = r.$$

4. Cartesian coordinates:

$$\operatorname{Jac}(u)(x,y,z) = \det \nabla u(x,y,z) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1.$$

Exercise 3.

1. The sketch:



The term $x^2 + y^2$ in the description of A suggests that it is appropriate to use polar coordinates: $(x,y) = (r\cos\theta, r\sin\theta)$ with $r \geq 0$ and $\theta \in [0, 2\pi]$,

$$x^2 + y^2 \leq 4 \iff r^2 \leq 4 \overset{r\geq 0}{\iff} 0 \leq r \leq 2.$$

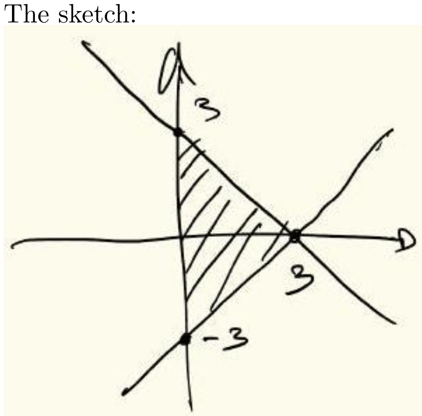
Thus,

$$A = \{(r\cos\theta, r\sin\theta) \mid r \in [0,2],\ \theta \in [0, 2\pi]\}$$

and we compute

$$\begin{aligned}
\int_A f(y)\, dy &= \int_0^2 \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr \\
&= \int_0^2 \int_0^{2\pi} \frac{1}{r} r\, d\theta\, dr \\
&= 4\pi.
\end{aligned}$$

2. The sketch:



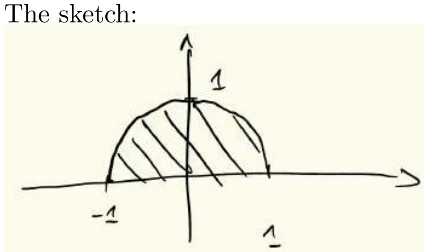
Here, we can use a Cartesian coordinate system directly:

$$\begin{aligned}
\int_A f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= \int_0^3 \int_{x-3}^{3-x} f(x,y)\mathrm{d}y\mathrm{d}x \\
&= \int_0^3 \int_{x-3}^{3-x} x^2 + \sin^3(y)\mathrm{d}y\mathrm{d}x \\
&= \int_0^3 (6-2x)x^2\mathrm{d}x + \int_0^3 \underbrace{\int_{x-3}^{3-x} \sin^3(y)\mathrm{d}y}_{=0}\,\mathrm{d}x.
\end{aligned}$$

The second integral is zero because the integrand is a odd function and the integration domain is symmetric with respect to the origin. Thus:

$$\begin{aligned}
\int_A f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= \int_0^3 6x^2 - 2x^3\mathrm{d}x \\
&= \left[2x^3 - \frac{1}{2}x^4\right]_{x=0}^{x=3} \\
&= \frac{27}{2}.
\end{aligned}$$

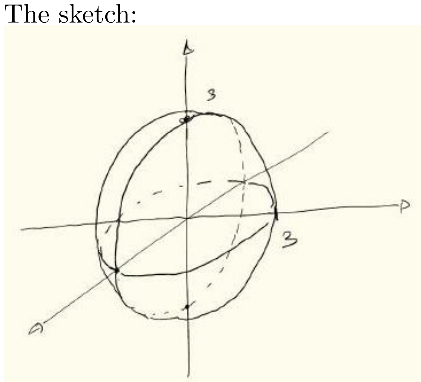
3. The sketch:



We can either use a Cartesian or a Polat coordinate system here. With Cartesian coordinates, we get:

$$\begin{aligned}\int_A f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} y\left(1+x^2\right)\mathrm{d}y\mathrm{d}x\\ &= \int_{-1}^{1}\left(1+x^2\right)\left[\frac{1}{2}y^2\right]_{y=0}^{y=\sqrt{1-x^2}}\mathrm{d}x\\ &= \frac{1}{2}\int_{-1}^{1}\left(1+x^2\right)\left(1-x^2\right)\mathrm{d}x\\ &= \frac{1}{2}\int_{-1}^{1} -x^4+1\mathrm{d}x\\ &= \frac{1}{2}\left[-\frac{1}{5}x^5+x\right]_{x=-1}^{x=1} = \frac{4}{5}.\end{aligned}$$

4. The sketch:



The term $x^2+y^2+z^2$ in the description of A suggests that it is appropriate to use spherical coordinates: $(x, y, z) = (r\cos\theta\sin\varphi,\ r\sin\theta\sin\varphi,\ r\cos\varphi)$ with $r \geq 0$, $\theta \in [0, 2\pi]$, and $\varphi \in [0, \pi]$,

$$x^2+y^2+z^2 \leq 9 \Longleftrightarrow r^2 \leq 9 \overset{r\geq 0}{\Longleftrightarrow} 0 \leq r \leq 3.$$

Thus,

$A = \{(r\cos\theta\sin\varphi,\ r\sin\theta\sin\varphi,\ r\cos\varphi) \mid r \in [0,3],\ \theta \in [0,2\pi],\ \varphi \in [0,\pi]\}$

and we compute

$$\int_A f(y)\,dy = \int_0^3 \int_0^{2\pi} \int_0^{\pi} r\sin\varphi\, r^2 \sin\varphi\, d\varphi\, d\theta\, dr$$
$$= 2\pi \int_0^3 r^3\, dr \int_0^{\pi} \sin^2\varphi\, d\varphi.$$

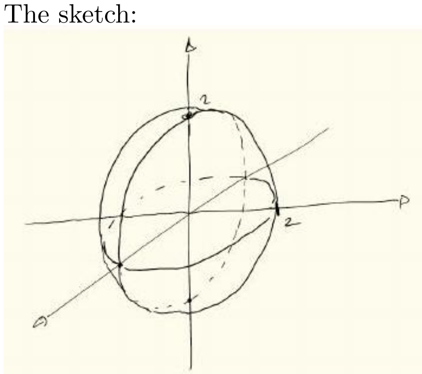
We recall that

$$\frac{d}{dt}\left[\tfrac{1}{2}\left(t - \sin(t)\cos(t)\right)\right] = \sin^2(t).$$

Therefore,

$$\int_A f(y)\,dy = 2\pi \left[\tfrac{1}{4}r^4\right]_{r=0}^{r=3} \left[\tfrac{1}{2}\left(\varphi - \sin(\varphi)\cos(\varphi)\right)\right]_{\varphi=0}^{\varphi=\pi}$$
$$= \tfrac{81}{4}\pi^2.$$

5. The sketch:



We employ the spherical coordinate system which leads to define the domain of integration as:

$A = \{(r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi) \mid r \in [0,2], \theta \in [0,2\pi], \varphi \in [0,\pi]\}.$

We obtain:

$$\int_A f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_0^2 \int_0^{2\pi} \int_0^{\pi} \arccos(\cos\varphi) r^2 \sin\varphi \mathrm{d}\varphi \mathrm{d}\theta \mathrm{d}r$$
$$= 2\pi \int_0^2 r^2 \mathrm{d}r \int_0^{\pi} \varphi \sin\varphi \mathrm{d}\varphi$$
$$= 2\pi \left[\frac{1}{3}r^3\right]_{r=0}^{r=2} \left([\varphi(-\cos\varphi)]_{\varphi=0}^{\varphi=\pi} + \int_0^{\pi} \cos\varphi \mathrm{d}\varphi \right)$$
$$= \frac{16}{3}\pi^2.$$

Exercise 4.

1. In a first step, we can show that:

$$\nabla(|x|) = \frac{x}{|x|}, \quad \forall x \in \mathbb{R}^n \backslash \{0\}.$$

A straightforward computation shows:

$$\frac{\partial}{\partial x_i}|x| = \frac{\partial}{\partial x_i}\left[\sqrt{\sum_{j=1}^{n} x_j^2}\right] = \frac{1}{2\sqrt{\sum_{j=1}^{n} x_j^2}}\frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n} x_j^2\right] = \frac{1}{2|x|}\sum_{j=1}^{n} \underbrace{\frac{\partial}{\partial x_i}\left[x_j^2\right]}_{=0 \text{ if } i\neq j \text{ and } =2x_i \text{ if } i=j} = \frac{x_i}{|x|}.$$

Consequently, we have:

$$\nabla\left(|x|^p\right) = p|x|^{p-1}\nabla(|x|) = p|x|^{p-2}x.$$

Remark: this function is well defined at $x = 0$ if $p \geq 2$.

2. By defining,

$$h_p : \mathbb{R}^n \to \mathbb{R} : u \mapsto h_p(u)$$
$$G : \mathbb{R}^n \to \mathbb{R}^n : v \mapsto G(v)$$
$$\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^n : t \mapsto \boldsymbol{\gamma}(t) = tx$$

we can rewrite the function f_p as a composition of these functions:

$$f_p = \frac{1}{p} h_p \circ \boldsymbol{G} \circ \boldsymbol{\gamma}.$$

Therefore, by applying the chain rule of differentiation, we have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f_p(t) &= \frac{1}{p}\sum_{i=1}^{n} \frac{\partial h_p}{\partial u_i}(\boldsymbol{G}(\boldsymbol{\gamma}(t)))\frac{dG_i \circ \boldsymbol{\gamma}}{dt}(t) \\
&= \frac{1}{p}\sum_{i=1}^{n} \frac{\partial h_p}{\partial u_i}(\boldsymbol{G}(\boldsymbol{\gamma}(t)))\sum_{j=1}^{n}\frac{\partial G_i}{\partial v_j}(\boldsymbol{\gamma}(t))\frac{d\gamma^j}{dt}(t)
\end{aligned}$$

Since,

$$\frac{\partial h_p}{\partial u_i}(u) = p|u|^{p-2}u_i, \quad \text{and} \quad \frac{d\gamma^j}{dt}(t) = x_j,$$

we get the final result:

$$\frac{\mathrm{d}}{\mathrm{d}t} f_p(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} |\boldsymbol{G}(tx)|^{p-2} G_i(tx)\frac{\partial G_i}{\partial v_j}(tx)x_j.$$