OPEN A

- Soit $a, b > 0$ donnés, $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 ;\ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1,\ x_3 = 0\}$

4pts Calculer l'aire de Σ

$\sigma(r, \theta) = (ar\cos\theta,\ br\sin\theta, 0) \qquad 0 < r < 1,\ 0 < \theta < 2\pi$ 1pt

$\frac{\partial\sigma}{\partial r} \wedge \frac{\partial\sigma}{\partial\theta} = \begin{pmatrix} a\cos\theta \\ b\sin\theta \\ 0 \end{pmatrix} \wedge \begin{pmatrix} -ar\sin\theta \\ br\cos\theta \\ 0 \end{pmatrix} = (0, 0, abr)$ 1pt

$\text{aire}(\Sigma) = \iint_\Sigma d\sigma = \int_0^{2\pi} d\theta \int_0^1 dr\ abr = \pi ab$ 1pt 1pt

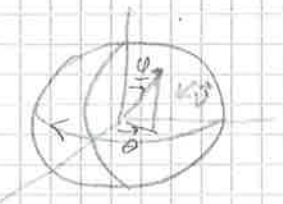
$\iint_\Sigma \vec{rot}\vec{F} \cdot \vec{ds} = \int_{\partial\Sigma} F \cdot \vec{d\ell}$

OPEN B

- Vérifier le théorème de Stokes pour $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 ;\ x_1^2 + x_2^2 + x_3^2 = 1,\ x_3 \geq 0\}$

10pts et $F(x_1 x_2 x_3) = (2x_1 - x_2,\ -x_2 x_3^2,\ -x_2^2 x_3)$

param Σ $\sigma(\theta, \varphi) = (\cos\theta\sin\varphi,\ \sin\theta\sin\varphi,\ \cos\varphi)$ 1pt



$0 < \theta < 2\pi \qquad 0 < \varphi < \pi/2$ 1pt

$\frac{\partial\sigma}{\partial\theta} \wedge \frac{\partial\sigma}{\partial\varphi} = \begin{pmatrix} -\sin\theta\sin\varphi \\ \cos\theta\sin\varphi \\ 0 \end{pmatrix} \wedge \begin{pmatrix} \cos\theta\cos\varphi \\ \sin\theta\cos\varphi \\ -\sin\varphi \end{pmatrix} = (-\cos\theta\sin^2\varphi,\ -\sin\theta\sin^2\varphi,\ -\sin\varphi\cos\varphi)$ vers l'intérieur

1pt 1pt (ou l'extérieur)

$\vec{rot}\vec{F} = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \wedge \begin{pmatrix} 2x_1 - x_2 \\ -x_2 x_3^2 \\ -x_2^2 x_3 \end{pmatrix} = (0, 0, +1)$ 1pt

$\iint_\Sigma \vec{rot}\vec{F} \cdot \vec{d\sigma} = \int_0^{2\pi} d\theta \int_0^{\pi/2} d\varphi\ (-\sin\varphi\cos\varphi) = -2\pi \left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/2} = -\pi$ 1pt (ou π si param vers l'ext)

$\partial\Sigma$: cercle centre O rayon 1 plan $x_3 = 0$ $\quad \gamma(t) = (\cos t, \sin t, 0)$

$\gamma'(t) = (-\sin t, \cos t, 0)$

1pt

mauvais sens, il faut changer le signe à la fin du calcul de l'intégrale

$\int_{\partial\Sigma} \vec{F} \cdot d\ell = \int_0^{2\pi} \vec{F}(\gamma(t)) \cdot \vec{\gamma}'(t)\, dt = \int_0^{2\pi} (2\cos t - \sin t, 0, 0) \cdot (-\sin t, \cos t, 0)\, dt$ 1pt

$= \int_0^{2\pi} (-2\cos t\sin t + \sin^2 t)\, dt = \left[(\cos(t))^2\right]_{t=0}^{t=2\pi} + \pi$

$= \pi$ 1pt

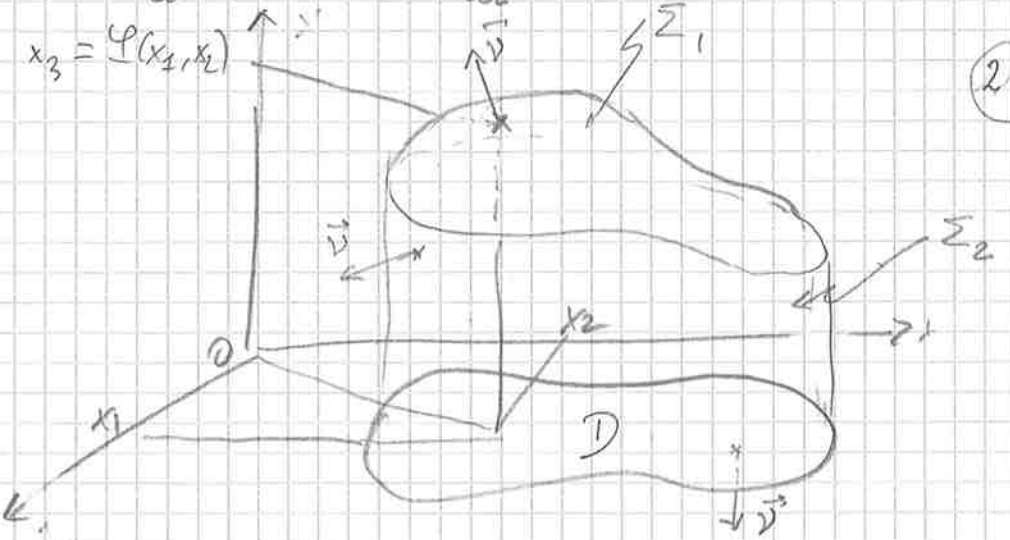
on change le signe et on a bien $\iint_\Sigma \vec{rot}\vec{F} \cdot d\sigma = \int_{\partial\Sigma} \vec{F} \cdot \vec{d\ell}$ 1pt

ou alors $\vec{\gamma}(t) = (\cos t, \sin t, 0)$ $2\pi \leq t \leq 0$

10pts OPEN C

Soit $D \subset \mathbb{R}^2$ un ouvert borné simplement connexe, soit $\varphi : D \to \mathbb{R}$ C^1 telle que $\varphi(x_1,x_2) > 0$ et soit $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3 ;\ (x_1,x_2) \in D,\ 0 < x_3 < \varphi(x_1,x_2)\}$

- Faire un dessin représenter Ω, $\partial\Omega$, $\vec{\nu}$ (on notera $\partial\Omega = D \cup \Sigma_1 \cup \Sigma_2$ où $\nu_3 = 0$ sur Σ_2)
- Montrer que $\iiint_\Omega \frac{\partial f}{\partial x_3} dx_1 dx_2 dx_3 = \iint_D \left(f(x_1,x_2,\varphi(x_1,x_2)) - f(x_1,x_2,0) \right) dx_1 dx_2$
- En déduire que $\iiint_\Omega \frac{\partial f}{\partial x_3} dx_1 dx_2 dx_3 = \iint_{\partial\Omega} f \nu_3 \, ds$



(2 pts)

$$\iiint_\Omega \frac{\partial f}{\partial x_3} dx_1 dx_2 dx_3 = \iint_D dx_1 dx_2 \int_0^{\varphi(x_1,x_2)} \frac{\partial f}{\partial x_3}(x_1 x_2 x_3)\, dx_3$$

$$= \iint_D \left(f(x_1,x_2,\varphi(x_1,x_2)) - f(x_1,x_2,0) \right) dx_1 dx_2$$ (1pt)

$\partial\Omega = D \cup \Sigma_1 \cup \Sigma_2$

param Σ_1 : $\sigma(x_1,x_2) = (x_1, x_2, \varphi(x_1,x_2))$ (1pt)

$$\frac{\partial\sigma}{\partial x_1} \wedge \frac{\partial\sigma}{\partial x_2} = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial\varphi}{\partial x_1} \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ \frac{\partial\varphi}{\partial x_2} \end{pmatrix} = \left(-\frac{\partial\varphi}{\partial x_1}, -\frac{\partial\varphi}{\partial x_2}, 1\right)$$ $\vec{\nu}$ vers le haut ok (1pt)

$$\left\| \frac{\partial\sigma}{\partial x_1} \wedge \frac{\partial\sigma}{\partial x_2} \right\| = \sqrt{1 + \left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi}{\partial x_2}\right)^2}$$

$$\iint_{\Sigma_1} f \nu_3 \, ds = \iint_D f(x_1,x_2,\varphi(x_1,x_2)) \frac{1}{\sqrt{1+}} \sqrt{1+} \; dx_1 dx_2$$

(1pt)

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param D : $\sigma(x_1, x_2) = (x_1, x_2, 0)$ (1pt)

$\frac{\partial \sigma}{\partial x_1} \wedge \frac{\partial \sigma}{\partial x_2} = (0, 0, 1)$ $\vec{\nu}$ vers le haut, on change le signe de l'intégrale à la fin du calcul (1pt)

$$\iint_D f \nu_3 \, ds = -\iint_D f(x_1, x_2, 0) \, dx_1 \, dx_2$$ (1pt)

$$\iint_{\Sigma_2} f \nu_3 \, ds = 0 \quad \text{car } \nu_3 = 0 \text{ sur } \Sigma_2.$$ (1pt)

OPEN D

Soit $f : \mathbb{C} \to \mathbb{C}$ définie par $f(z) = z^2$. Pour $z = x+iy$, on note $f(z) = \phi(x,y) + i\Psi(x,y)$ où ϕ et $\Psi : \mathbb{R}^2 \to \mathbb{R}$.

- Montrer que f est holomorphe dans $\mathbb{C}$, $\Delta\phi = 0$, $\vec{\nabla}\varphi \cdot \vec{\nabla}\Psi = 0$
- On pose $\vec{u} = \underbrace{\vec{\nabla}\Psi}_{=(u,v)}$, représenter dans $D = \{(x,y) \in \mathbb{R}^2 ; y > 0\}$

  les lignes $\phi = cste$, $\Psi = cste$, le champ $\vec{u}$
- $\leftarrow$ Expliciter les trajectoires (les lignes $x(t)$ et $y(t)$ telles que $x'(t) = u(x(t), y(t))$ et $y'(t) = v(x(t), y(t))$)
- $\leftarrow$ Montrer que si $x(0) > 0$ et $y(0) = 0$ alors $\lim_{t\to\infty} x(t) = +\infty$ et $y(t) = 0$
- $\leftarrow$ $x(0) = 0$ et $y(t) > 0$ —— $x(t) = 0$ et $\lim_{t\to\infty} y(t) = 0$

10 pts

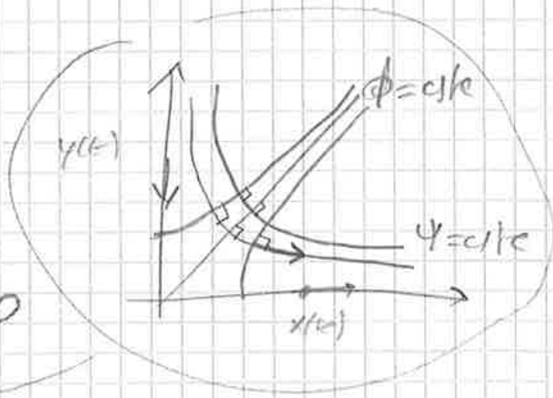
1pt · $f(z) = (x+iy)^2 = \underbrace{x^2 - y^2}_{\phi(x,y)} + i\underbrace{2xy}_{\Psi(x,y)}$

1pt f holomorphe car $\phi, \Psi : \mathbb{R}^2 \to \mathbb{R}$ sont $\mathcal{C}^1$ et $\frac{\partial\phi}{\partial x} = 2x = \frac{\partial\Psi}{\partial y}$ et $\frac{\partial\phi}{\partial y} = -2y = -\frac{\partial\Psi}{\partial x}$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial\Psi}{\partial y}$$

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial\Psi}{\partial x}\right)$$

1pt $\Delta\phi = 0$



1pt $\nabla\phi \cdot \nabla\Psi = \frac{\partial\phi}{\partial x}\frac{\partial\Psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\Psi}{\partial y} = \frac{\partial\phi}{\partial x}\left(-\frac{\partial\phi}{\partial y}\right) + \frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x} = 0$

- $\leftarrow$ $u(x,y) = \frac{\partial\phi}{\partial x} = 2x$ $v(x,y) = \frac{\partial\phi}{\partial y} = -2y$

3pts (1pt pour $\phi = cste$, 1pt $\Psi = cste$, 1pt $\vec{u}$)

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- $x'(t) = 2x(t)$, $y'(t) = -2y(t)$, $x(t) = x(0)e^{2t}$ (1pt), $y(t) = y(0)e^{-2t}$ (1pt)

Si $y(0) = 0$ $y(t) = 0$ et si $x(t) > 0$ $\lim_{t\to\infty} x(t) = +\infty$

Si $x(0) = 0$ $x(t) = 0$ et si $y(t) > 0$ $\lim_{t\to\infty} y(t) = 0$ (1pt)