EPFL – Automne 2025	Pr. M. Picasso
Analyse III – PH	**Corrigé**
Série 1	11 septembre 2025

Exercice 1.

1. c.f. cours
2. c.f. cours
3. En utilisant la *chain rule* et la définition de grad(·), on a

$$\begin{aligned}
\operatorname{grad}(fg) &= (\frac{\partial(fg)}{\partial x_1}, \frac{\partial(fg)}{\partial x_2}, \dots, \frac{\partial(fg)}{\partial x_n}) \\
&= (\frac{\partial f}{\partial x_1}g + \frac{\partial g}{\partial x_1}f, \frac{\partial f}{\partial x_2}g + \frac{\partial g}{\partial x_2}f, \dots, \frac{\partial f}{\partial x_n}g + \frac{\partial g}{\partial x_n}f) \\
&= g(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}) + f(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots, \frac{\partial g}{\partial x_n}) \\
&= g \operatorname{grad}(f) + f \operatorname{grad}(g).
\end{aligned}$$

4. En utilisant la *chain rule* et la définition de grad(·) et div(·), on a

$$\begin{aligned}
\operatorname{div}(fF) &= \sum_{i=1}^{n} \frac{\partial((fF)_i)}{\partial x_i} \\
&= \sum_{i=1}^{n} \left(f\frac{\partial F_i}{\partial x_i} + F_i \frac{\partial f}{\partial x_i} \right) \\
&= f \operatorname{div}(F) + F \cdot \operatorname{grad}(f).
\end{aligned}$$

5. En utilisant la définition du rotationnel, on obtient

$$\begin{aligned}
\operatorname{rot}(\operatorname{rot}(F)) &= \operatorname{rot}\,((\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2})) \\
&= (\frac{\partial^2 F_2}{\partial x_1 \partial x_2} - \frac{\partial^2 F_1}{\partial x_2^2} - \frac{\partial^2 F_1}{\partial x_3^2} + \frac{\partial^2 F_3}{\partial x_1 \partial x_3}, \frac{\partial^2 F_3}{\partial x_2 \partial x_3} - \frac{\partial^2 F_2}{\partial x_3^2} - \frac{\partial^2 F_2}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial x_1 \partial x_2}, \\
&\quad \frac{\partial^2 F_1}{\partial x_3 \partial x_1} - \frac{\partial^2 F_3}{\partial x_1^2} - \frac{\partial^2 F_3}{\partial x_2^2} + \frac{\partial^2 F_2}{\partial x_2 \partial x_3}) \\
&= -\Delta F + (\frac{\partial^2 F_1}{\partial x_1^2}, \frac{\partial^2 F_2}{\partial x_2^2}, \frac{\partial^2 F_3}{\partial x_3^2}) + (\frac{\partial^2 F_2}{\partial x_1 \partial x_2} + \frac{\partial^2 F_3}{\partial x_1 \partial x_3}, \frac{\partial^2 F_3}{\partial x_2 \partial x_3} + \frac{\partial^2 F_1}{\partial x_1 \partial x_2}, \frac{\partial^2 F_1}{\partial x_3 \partial x_1} + \frac{\partial^2 F_2}{\partial x_2 \partial x_3}) \\
&= -\Delta F + (\frac{\partial}{\partial x_1}(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}), \frac{\partial}{\partial x_2}(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}), \frac{\partial}{\partial x_3}(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3})) \\
&= -\Delta F + \operatorname{grad}(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}) \\
&= -\Delta F + \operatorname{grad}(\operatorname{div}(F)).
\end{aligned}$$

6. En utilisant la définition du rotationnel et du produit vectoriel ($\wedge$), on obtient

$$
\begin{aligned}
\text{rot}(fF) &= (\frac{\partial (fF)_3}{\partial x_2} - \frac{\partial (fF)_2}{\partial x_3}, \frac{\partial (fF)_1}{\partial x_3} - \frac{\partial (fF)_3}{\partial x_1}, \frac{\partial (fF)_2}{\partial x_1} - \frac{\partial (fF)_1}{\partial x_2}) \\
&= (f\frac{\partial F_3}{\partial x_2} + F_3\frac{\partial f}{\partial x_2} - f\frac{\partial F_2}{\partial x_3} - F_2\frac{\partial f}{\partial x_3}, f\frac{\partial F_1}{\partial x_3} + F_1\frac{\partial f}{\partial x_3} - f\frac{\partial F_3}{\partial x_1} - F_3\frac{\partial f}{\partial x_1}, \\
&\quad f\frac{\partial F_2}{\partial x_1} + F_2\frac{\partial f}{\partial x_1} - f\frac{\partial F_1}{\partial x_2} - F_1\frac{\partial f}{\partial x_2}) \\
&= f(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}) + (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}) \wedge (F_1, F_2, F_3) \\
&= f\,\text{rot}(F) + \text{grad}(f) \wedge (F).
\end{aligned}
$$

Exercice 2.

Les points 1. et 2. sont vrais.

Le point 3. est faux. En effet, si $T(x_1, x_2, x_3) = e^{\frac{x_2^2}{2}} + C$ et $v = (0, x_2, 0)$, on a

$$
-\text{div}(\text{grad}(T)) + \text{div}(Tv) = -e^{\frac{x_2^2}{2}}(1 + x_2^2) + e^{\frac{x_2^2}{2}}(1 + x_2^2) + C.
$$

T ne satisfait donc l'équation de la chaleur que si $C = 0$.

Exercice 3.

1. Vrai.
2. Faux (la condition $\text{div}(v) = 0$ n'est pas vérifiée); il faudrait définir $v(x_1, x_2, x_3) = (\frac{x_2^2 + x_3^2}{4}, 0, 0)$ pour que v, p satisfassent (2)-(5).

Exercice 4.

1. Vrai.
2. Faux; les équation (6)-(8) peuvent s'écrire $-\mu\,\text{rot}(\text{rot}(u)) + (\lambda + 2\mu)\,\text{grad}(\text{div}(u)) + f = 0$.
3. Vrai.
4. Vrai.
5. Vrai.