MATH-111(en) FALL 2025
Linear Algebra
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SOLUTIONS for Homework 9

Ex 9.1 (Column, row and kernels)
Find the dimensions of the column space, row space, and kernel of the following matrix.

$$B = \begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix}$$

Solution:
We do row reduction:

$$\begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ -2 & -5 & 8 & 0 & -17 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 20 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Because the echelon form has three pivots, the column space has dimension 3. Because the system $Bx = 0$ has two free variables, the kernel has dimension 2 (alternatively one can use the rank theorem).
In the course we proved that the dimension of the row space equals the dimension of the column space. Hence the row space is also three-dimensional.

Ex 9.2 (A subspace)
Find out the dimension of the subspace H defined as:

$$H = \left\{ x \text{ in } \mathbb{R}^4 \text{ such that } x = \begin{pmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{pmatrix} \text{ where } a, b, c \text{ and } d \text{ are real scalars} \right\}.$$

Solution:
By definition, H is a subspace of $\mathbb{R}^4$ defined as $\text{Span}\{v_1, v_2, v_3, v_4\}$ with:

$$v_1 = \begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 6 \\ 0 \\ -2 \\ 0 \end{pmatrix} \quad \text{and} \quad v_4 = \begin{pmatrix} 0 \\ 4 \\ -1 \\ 5 \end{pmatrix}.$$

One can see that $v_3 = -2v_2$, which yields that $H = \text{Span}\{v_1, v_2, v_4\}$.
Then we check that those three vectors are linearly independent (you can for instance compute an echelon form of the matrix having v_1, v_2 and v_4 as columns and which has three pivot columns), to deduce that H has dimension 3.

Ex 9.3 (Row equivalent matrices)
Consider

$$A = \begin{pmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Show that the matrices A and B are row equivalent.
Deduce

- the rank of A and $\dim \text{Ker}\, A$
- a basis for each of the subspaces $\text{Col}\, A$, $\text{Row}\, A$, and $\text{Ker}\, A$.

Solution:
We see that the reduced echelon forms of both matrices are the same, thus they are equivalent. (Here it is faster to start row reduction with A and one obtains B after adding the first row to the second one, then subtracting 5 times the first row from the third one and finally eliminating the last row with the help of the new second row).
By looking at the matrix B one can notice that:

- There are two pivot columns which gives $\text{rank}\, B = \text{rank}\, A = 2$ (the rank is the number of pivot columns). A basis of $\text{Col}\, A$ is furthermore given by the first two columns of A, which correspond to the pivot columns of its echelon form. By the rank theorem we find $\dim \text{Ker}\, A = n - \text{rank}\, A = 4 - 2 = 2$.
- As discussed in the first part, a basis of $\text{Col}\, A$ is given by $\left\{ \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix} \right\}$.

  As A and B are row equivalent, $\text{Row}\, A = \text{Row}\, B$ and we know that the non-zero rows of any echelon form form a basis for the row space. Hence a basis of $\text{Row}\, A$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 5 \\ -6 \end{pmatrix} \right\}$.

  The equation $Ax = 0$ is equivalent to $Bx = 0$. Using the parametric vector form for the solution, we know that a basis of $\text{Ker}\, A$ is given for instance by $\left\{ \begin{pmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Ex 9.4 (Relating A and A^T in terms of linear systems)
Consider a matrix $A \in \mathbb{R}^{m \times n}$. Among the spaces $\text{Row}\, A$, $\text{Col}\, A$, $\text{Ker}\, A$, $\text{Row}\, A^T$, $\text{Col}\, A^T$ and $\text{Ker}\, A^T$, find out which are subspaces of $\mathbb{R}^n$, and which are subspaces of $\mathbb{R}^m$.
Then justify the following statements:

1. $\dim \text{Row}\, A + \dim \text{Ker}\, A = n$ (number of columns in A).

2. $\dim \operatorname{Col} A + \dim \operatorname{Ker} A^T = m$ (number of rows in A).

3. $Ax = b$ has a solution for every b in $\mathbb{R}^m$ if and only if $A^T x = 0$ only admits the trivial solution.

Solution:
$\operatorname{Row} A = \operatorname{Col} A^T$ and $\operatorname{Ker} A$ are subspaces of $\mathbb{R}^n$; $\operatorname{Col} A = \operatorname{Row} A^T$ and $\operatorname{Ker} A^T$ are subspaces of $\mathbb{R}^m$.

1. We know that $\dim \operatorname{Row} A = \dim \operatorname{Col} A = \operatorname{rank}(A)$ so from the rank theorem: $\dim \operatorname{Row} A + \dim \operatorname{Ker} A = n$.

2. All you have to do is to write the previous statement replacing A by A^T in it, and note that $\operatorname{Row} A^T = \operatorname{Col} A$.

3. Saying that $Ax = b$ always has a solution is equivalent to affirm that $\dim \operatorname{Col} A = m$, which from the previous point gives $\dim \operatorname{Ker} A^T = 0$. This last statement is true if and only if the equation $A^T x = 0$ only admits the trivial solution.

Ex 9.5 (Change of basis matrices)
Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^3$, and consider the following basis:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

Find the change of basis matrices $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ and $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$.

Solution:
According to the lecture the matrix $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ is the matrix with columns $[b_1]_\mathcal{E}, [b_2]_\mathcal{E}, [b_3]_\mathcal{E}$, where b_1, b_2 and b_3 are the elements of the basis $\mathcal{B}$. Hence we need to find the standard coordinates of the basis vectors in $\mathcal{B}$. Without calculation, we see have immediately that

$$\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -3 \\ 0 & 1 & 1 \end{pmatrix}.$$

As seen in the lecture, the matrix $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$ is the inverse of $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$. Let us compute the inverse:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -4 & -3 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -4 & -3 & -2 & 1 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & -3 \\ 0 & 0 & 1 & -2 & 1 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 6 \\ 0 & 1 & 0 & 2 & -1 & -3 \\ 0 & 0 & 1 & -2 & 1 & 4 \end{array}\right)$$

So $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P} = \begin{pmatrix} -3 & 2 & 6 \\ 2 & -1 & -3 \\ -2 & 1 & 4 \end{pmatrix}$.

Ex 9.6 (Changing coordinates)
Consider the following bases of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \qquad \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ Then determine $[v]_{\mathcal{C}}$ for $[v]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution:
As discussed in the course, the way to compute this is with the following row reduction (where B and C are the matrices with the columns from $\mathcal{B}$ and $\mathcal{C}$ in the given order):

$$(\ C \mid B\ ) \longrightarrow \left(\ I \mid \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right).$$

Let us see briefly recall why. We want the matrix P such that $[w]_{\mathcal{C}} = P \cdot [w]_{\mathcal{B}}$ for every vector w. In particular, it should work for the vectors b_1, b_2, b_3 of $\mathcal{B}$. Note that $[b_i]_{\mathcal{B}} = e_i$, and that $P \cdot e_i = p_i$, where p_i is the i-th column of P. So for instance, P should satisfy

$$[b_1]_{\mathcal{C}} = P \cdot [b_1]_{\mathcal{B}} = P \cdot e_1 = p_1.$$

So we want the vectors p_i such that

$$[b_1]_{\mathcal{C}} = p_1, \quad [b_2]_{\mathcal{C}} = p_2, \quad [b_3]_{\mathcal{C}} = p_3.$$

In other words, we want to represent each vector b_i in the basis $\mathcal{C}$. Interpreting each b_i as its coordinates in the standard basis and C as the change-of-coordinates matrix $P_{\mathcal{C}}$, we see that p_i is the solution of $Cx = b_i$. We can solve these three systems at the same time by row reducing $(C|B) \to (I|P)$.
So, we do that reduction for the given bases $\mathcal{B}$ and $\mathcal{C}$:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -4 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad \Rightarrow \quad \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To get $[v]_{\mathcal{C}}$ we just have to multiply by the obtained change of basis matrix:

$$[v]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot [v]_{\mathcal{B}} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

To verify the result, we can compare what both representations would give in the standard basis:

$$v = 1 \cdot b_1 + 1 \cdot b_2 + 1 \cdot b_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1 \cdot c_1 + 0 \cdot c_2 + 1 \cdot c_3.$$

Ex 9.7 (More basis changes)
Let $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ be two bases of a vector space V. Assume that $b_1 = 6c_1 - 2c_2$ and $b_2 = 9c_1 - 4c_2$.

(a) Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

(b) Find $[x]_{\mathcal{C}}$ for $x = -3b_1 + 2b_2$. Use the result from (a).

Let $\mathcal{A} = \{a_1, a_2\}$ and $\mathcal{D} = \{d_1, d_2\}$ be two bases of $\mathbb{R}^2$.

$$a_1 = \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \quad a_2 = \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \quad d_1 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \quad d_2 = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

(c) Find the change of basis matrix $\underset{\mathcal{D}\leftarrow\mathcal{A}}{P}$

(d) Find the change of basis matrix $\underset{\mathcal{A}\leftarrow\mathcal{D}}{P}$

Solution:

(a) The change of basis $\mathcal{B} \to \mathcal{C}$ matrix has in its columns the coordinates of the basis vectors of the basis $\mathcal{B}$, in the basis $\mathcal{C}$:

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{pmatrix} 6 & 9 \\ -2 & -4 \end{pmatrix}$$

(b) The equation $x = -3b_1 + 2b_2$ implies that $[x]_{\mathcal{B}} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$. To find $[x]_{\mathcal{C}}$, all you have to do is to use the change of basis matrix:

$$[x]_{\mathcal{C}} = \begin{pmatrix} 6 & 9 \\ -2 & -4 \end{pmatrix} [x]_{\mathcal{B}} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

(c) To find the change of basis $\mathcal{A} \to \mathcal{D}$ matrix one needs to write the coordinates of the vectors a_1 and a_2 in the basis $\mathcal{D}$. That is find the real numbers x_1 and x_2 such as $a_1 = x_1 d_1 + x_2 d_2$ and the real numbers y_1 and y_2 such as $a_2 = y_1 d_1 + y_2 d_2$.

In order to find those, one can simply solve the two systems:

$$[d_1\ d_2] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a_1 \quad \text{and} \quad [d_1\ d_2] \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = a_2$$

The solution is $x_1 = -3$, $x_2 = -5$, $y_1 = 1$, $y_2 = 2$ which gives:

$$\underset{\mathcal{D}\leftarrow\mathcal{A}}{P} = \begin{pmatrix} -3 & 1 \\ -5 & 2 \end{pmatrix}$$

(d) To find $\underset{\mathcal{A}\leftarrow\mathcal{D}}{P}$ we need to invert $\underset{\mathcal{D}\leftarrow\mathcal{A}}{P}$:

$$\underset{\mathcal{A}\leftarrow\mathcal{D}}{P} = \begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix}$$

Ex 9.8 (Basis change for polynomials)
In $\mathbb{P}_2$, find out the change of base matrix from the basis $\mathcal{B} = \{1-2t+t^2, 3-5t+4t^2, 2t+3t^2\}$ to the standard basis $\mathcal{C} = \{1, t, t^2\}$. Then write out the coordinates of the vector $x(t) = -1+2t$ in the basis $\mathcal{B}$.

Solution:
The change of basis $\mathcal{B} \to \mathcal{C}$ matrix is the matrix having for columns the coordinates of the basis vectors of $\mathcal{B}$ in the standard basis $\mathcal{C}$:

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{pmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{pmatrix}$$

The coordinates of the vector $x(t) = -1+2t$ in the standard basis are: $[x]_{\mathcal{C}} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$.

The coordinates $[x]_{\mathcal{B}}$ of the vector in the basis $\mathcal{B}$ satisfy: $\begin{pmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{pmatrix} [x]_{\mathcal{B}} = [x]_{\mathcal{C}}$.

All you have to do is to solve the linear system corresponding to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ -2 & -5 & 2 & 2 \\ 1 & 4 & 3 & 0 \end{array}\right] \text{ to finally find } [x]_{\mathcal{B}} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}.$$

Ex 9.9 (The trace of a matrix as linear map)
Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. We define the trace of A by $\mathrm{Tr}(A) = a_{11} + \ldots + a_{nn}$, i.e., the sum of all diagonal elements.

a) Show that the map $\mathrm{Tr} : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a linear map.

b) Consider the case $n = 2$ and the ordered basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

while on $\mathbb{R}$ we consider the standard basis $\mathcal{Q} = \{1\}$. Compute the matrix B such that $[\mathrm{Tr}(A)]_{\mathcal{Q}} = B[A]_{\mathcal{B}}$ for all $A \in \mathbb{R}^{2\times 2}$.

Solution:
a) Let $A, B \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$. According to the rules for matrix calculus the coefficients of the matrix $\lambda A + B$ are given by $\lambda a_{ij} + b_{ij}$. Therefore we have

$$\mathrm{Tr}(\lambda A + B) = \sum_{i=1}^{n} \lambda a_{ii} + b_{ii} = \lambda \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \lambda \mathrm{Tr}(A) + \mathrm{Tr}(B).$$

Thus the taking the trace is a linear operation.

b) According to the lecture, we have to compute the $\mathcal{Q}$-coordinates of $\mathrm{Tr}(b)$ for all elements in the basis $\mathcal{B}$. In this setting the $\mathcal{Q}$-coordinates are just the values of trace. The first matrix has trace 1, the second 0, the third 0 and the last one 1. Therefore $B = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$.

Ex 9.10 (Finding the matrix of a linear transformation - warm up)

(a) Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be defined by

$$T(a_0 + a_1x + a_2x^2) = a_0 + a_2 + (2a_1 + a_2)x + (2a_1 + a_2)x^3.$$

Find a basis for $\mathrm{Ker}(T)$. Moreover: is $p(x) = 5x^2 - 5$ in $\mathrm{Im}(T)$? Is it in $\mathrm{Ker}(T)$?

(b) Let $T : \mathbb{P}_3 \to \mathbb{R}^{2\times 3}$ be defined by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = \begin{pmatrix} a_1 + a_2 & a_2 + a_3 & a_3 \\ a_2 + a_3 & 0 & a_0 \end{pmatrix}$$

Find the matrix A of T relative to the standard bases of $\mathbb{P}_3$ and $\mathbb{R}^{2\times 3}$. Then find a basis for $\mathrm{Ker}(A)$, $\mathrm{Col}(A)$ and $\mathrm{Row}(A)$. Also find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Solution:

(a) The matrix of T with respect to the standard bases $\mathcal{B} = \{1, x, x^2\}$ for $\mathbb{P}_2$ and $\mathcal{C} = \{1, x, x^2, x^3\}$ for $\mathbb{P}_3$ is:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

You can <u>either</u> find this matrix by computing its columns which are:

$$[T(1)]_{\mathcal{C}} = [1]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad [T(x)]_{\mathcal{C}} = [2x + 2x^3]_{\mathcal{C}} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \quad [T(x^2)]_{\mathcal{C}} = [1 + x + x^3]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

<u>Or</u>, you can find the matrix A by solving the equation $[T(a_0 + a_1x + a_2x^2 + a_3x^3)]_{\mathcal{C}} = A([a_0 + a_1x + a_2x^2 + a_3x^3]_{\mathcal{B}})$ for A. Which means finding a 3×2 matrix A such that

$$\begin{pmatrix} a_0 + a_2 \\ 2a_1 + a_3 \\ 0 \\ 2a_1 + a_3 \end{pmatrix} = A \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}.$$

(It is not hard to come up with the correct matrix in this easy example.)

Once you have A, compute its reduced row echelon form, which is

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, solving the homogenous equation $Ax = 0$, we find $\mathrm{Ker}(A) = \left\{ \begin{pmatrix} -\lambda \\ -\frac{\lambda}{2} \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}$ and a basis of $\mathrm{Ker}(A)$ is $\mathcal{B}_{\mathrm{Ker}(A)} = \left\{ \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \right\}$.

Finally, for $[p(x)]_{\mathcal{B}} = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$ we have $p(x) = -1 - \frac{1}{2}x + x^2$.

Hence $\mathcal{B}_{\text{Ker}(T)} = \{-1 - \frac{1}{2}x + x^2\}$ is a basis of $\text{Ker}(T)$

By directly computing $T(5x^2 - 5) = 5x^3 + 5x \neq 0$, we conclude that $5x^2 - 5 \notin \text{Ker}(T)$.

By observing that the coefficient in front of x^2 in $T(p(x))$ is always 0, there cannot exists a $p(x) \in \mathbb{P}_2$ such that $T(p(x)) = 5x^2 - 5$. Thus, $5x^2 - 5 \notin \text{Im}(T)$.

(b) The matrix of T with respect to the standard basis $\mathcal{B} = \{1, x, x^2, x^3\}$ of $\mathbb{P}_3$ and the standard basis

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

of $\mathbb{R}^{2\times 3}$ is:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(You can find A by either method mentioned in part (a) above.)

Once you have A, compute its reduced row echelon form, which is

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So we have a bases for $\text{Ker}(A)$, $\text{Col}(A)$ and $\text{Row}(A)$: $\mathcal{B}_{\text{Ker}(A)} = \emptyset$,

$$\mathcal{B}_{\text{Col}(A)} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

and $\mathcal{B}_{\text{Row}(A)} = \left\{ \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \right\}$.

And bases for $\text{Ker}(T)$ and $\text{Im}(T)$ are:

$$\mathcal{B}_{\text{Ker}(T)} = \emptyset \quad \text{(or, you could also say: Ker}(T) \text{ does not have a basis)}$$

$$\mathcal{B}_{\text{Im}(T)} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\}.$$

Ex 9.11 (Finding the matrix of a linear transformation)

Let $T : \mathbb{R}^{2\times 2} \to \mathbb{P}_5$ be defined by

$$T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = 2x^5 - 3x^4 + 5x, \quad T\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = -x^2 + x + 1,$$

$$T\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = -x^2 + x + 1, \quad T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = -2x^4 + x^3 - x^2 + 1$$

Find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

Solution Denote $\mathcal{S}$ for the standard basis of $\mathbb{R}^{2\times 2}$, namely

$$\mathcal{S} = \left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}.$$

Inspired by the definition of T, we define the following basis for $\mathbb{P}_5$:

$$\mathcal{B} = \{p_1(x), p_2(x), p_3(x), p_4(x), p_5(x), p_6(x)\}$$

where we define

$$\begin{aligned}
p_1(x) &= 2x^5 - 3x^4 + 5x,\\
p_2(x) &= -x^2 + x + 1,\\
p_3(x) &= -2x^4 + x^3 - x^2 + 1,\\
p_4(x) &= x^3,\\
p_5(x) &= x,\\
p_6(x) &= 1.
\end{aligned}$$

Notice that we came up with this basis as follows: We observe that among the images of elements of $\mathcal{S}$ under T gives three independent polynomials. We choose these as the first three elements of our basis p_1, p_2, p_3. Then we add three more polynomials p_4, p_5, p_6 that are as simple as possible and that actually turn the whole collection $p_1, ..., p_6$ into a basis.

(Formal verification that $\mathcal{B}$ is actually a basis: identify P_5 with $\mathbb{R}^6$ (via the standard basis), then $p_1, ..., p_6$ are the columns of the the following matrix

$$\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 1\\
5 & 1 & 0 & 0 & 1 & 0\\
0 & -1 & -1 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0\\
-3 & 0 & -2 & 0 & 0 & 0\\
2 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

By reordering the columns, we can easily make the above matrix into an upper triangular matrix with non-zero coefficients on the diagonal. Thus, the matrix has full rank (you can observe this by either taking the determinant or making the matrix in to its RREF). Hence $\mathcal{B}$ is a basis of $\mathbb{P}_5$.)

By the same method as the previous exercise, we find the matrix A corresponding to T with respect to $\mathcal{S}$ and $\mathcal{B}$ is

$$A = \begin{pmatrix}
1 & 0 & 0 & 0\\
0 & 1 & 1 & 0\\
0 & 0 & 0 & 1\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0
\end{pmatrix}.$$

(Note how our clever choice of basis automatically makes the matrix A be in RREF!)

Thus, the polynomials corresponding to the pivots A form a basis of $\mathrm{Im}(T)$, namely

$$\mathcal{B}_{\mathrm{Im}(T)} = \{p_1(x), p_2(x), p_3(x)\}.$$

On the other hand, denoting $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, by solving the homogeneous system $A[x]_{\mathcal{S}} = 0$, we see that $a = d = 0$ and $b = -c$. Thus, we have $\mathcal{B}_{\mathrm{Ker}(T)} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$.

Ex 9.12 (True/False questions)

In the following, let A be an $m \times n$ matrix and $\mathcal{B}, \mathcal{C}$ bases of a vector space V. Decide whether the following statements are always true or if they can be false.

(i) $\mathrm{Row}(A) = \mathrm{Col}(A^T)$.

(ii) $\dim \mathrm{Row}(A) = \dim \mathrm{Col}(A)$.

(iii) $\dim \mathrm{Row}(A) + \dim \mathrm{Ker}(A) = n$.

(iv) There is a 6×9 matrix B such that $\dim \mathrm{Ker}(B) = 2$.

(v) If a set $\{v_1, \ldots, v_p\}$ spans a finite-dimensional vector space V and if T is a set of more than p vectors in V, then T is linearly dependent.

(vi) The only three-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.

(vii) If B is any echelon form of A, and if B has three nonzero rows, then the first three rows of A form a basis for $\mathrm{Row}\, A$.

(viii) The dimension of the kernel of A is the number of columns of A that are *not* pivot columns.

(ix) The row space of A^T is the same as the column space of A.

(x) The columns of the change-of-coordinates matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ are $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$.

(xi) If $V = \mathbb{R}^n$ and $\mathcal{C}$ is the *standard* basis V, then $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ is the same as the change-of-coordinates matrix $P_{\mathcal{B}}$ introduced earlier.

Solution:

(i) **True.** The rows of A are the columns of A^T.

(ii) **True.** Both are equal to the number of pivots in an echelon form of A.

(iii) **True.** This is essentially the rank theorem. ($\dim \mathrm{Row}(A)$ equals the number of pivots in an echelon form of A, and $\dim \mathrm{Ker}(A)$ equals the number of columns in an echelon form that do not have a pivot (because each such column gives a free variable), so the sum of these is the number of columns.)

(iv) **False.** By the rank theorem, we would have $\dim \mathrm{Col}(B) = 9 - 2 = 7$. But $\mathrm{Col}(B)$ is a subspace of $\mathbb{R}^6$, which has dimension 6, and subspaces cannot have higher dimension than the vector space they are in.

(v) **True.** Since V is spanned by p elements, a basis of, i.e. a maximal linearly independent subset in V has at most p elements.

(vi) **True.** Given 3 linearly independent vectors in $\mathbb{R}^3$, the reduced echelon form of the matrix A that has those vectors as columns will be the identity matrix, meaning that it is invertible and hence in particular that its column space is all of $\mathbb{R}^3$.

(vii) **False.** In general, the first 3 rows of B form a basis for Row A in this situation.

(viii) **True.**

(ix) **True.** The rows of A^T are the columns of A.

(x) **False.** The columns are $\mathcal{C}$-*coordinate vectors of the vectors in* $\mathcal{B}$.

(xi) **True.**