MATH-111(en) FALL 2025
Linear Algebra

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SOLUTIONS for Homework 8

Ex 8.1 (A family of bases)
Find all $b \in \mathbb{R}$ such that the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ b \\ 0 \end{pmatrix}$$

form a basis of $\mathbb{R}^3$.

Solution:
We need to find all b such that the three vectors are linearly independent and span $\mathbb{R}^3$. Equivalently, we want b such that the matrix with those three vectors as columns has an echelon form with three pivots. So:

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & b \\ 1 & 3 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & b-1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & -3 & b-1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & b-4 \end{pmatrix}$$

For $b = 4$, the bottom right entry is zero, so there are only two pivots, and the three vectors do not form a basis.
For $b \neq 4$, there are three pivots, so the vectors form a basis.

Ex 8.2 (Basis or not?)
Determine if $\{1 + t^2, 1 - t, 2 - 4t + t^2, 6 - 18t + 9t^2 - t^3\}$ is a basis for $\mathbb{P}_3 = \{\text{degree } \leq 3 \text{ polynomials in } t\}$.

Solution:
According to Corollary 4.14 we can write each polynomial as a vector in $\mathbb{R}^4$ with respect to the monomial basis and use row reduction to determine if these vectors are linearly independent.

$$\begin{pmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 1 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 3 & 21 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Since this echelon form has four pivots, the four vectors are linearly independent and span $\mathbb{R}^4$. Applying Corollary 4.14, the four polynomials form a basis for $\mathbb{P}_3$.

Ex 8.3 (Bases of column and kernel)
Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

(a) Find a basis for the column space of A.

(b) Find a basis for the kernel of A.

(c) What are the respective dimensions of the image and kernel of A?

Solution:

(a) We do column reduction on the matrix A:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Now A is in reduced column form and so its non-zero columns form a basis of $\mathrm{Col}(A)$

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix} \right\} \quad \text{is a basis of } \mathrm{Col}(A).$$

In particular, $\mathrm{Col}(A)$ has dimension 2.

<u>*Alternative solution*</u> *(via row-reduction but be careful!)*
We first do row reduction find the reduced row echelon form of A.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We observe that the pivots are in the first and second column. Hence, a basis for the column space is given by the first and second column of the original(!!!) matrix A, i.e.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix} \right\} \quad \text{is a basis of } \mathrm{Col}(A).$$

In particular, $\mathrm{Col}(A)$ has dimension 2.

(By the way: For the matrix A in this problem, one can also see directly from the original form of A that these two columns form a basis, because they are obviously independent and the third and fourth are copies of them.)

(b) Looking at the reduced echelon form, we see that the solution set of $Ax = 0$ is described by $x_1 = -x_3$ and $x_2 = -x_4$. The parametric vector form for this solution set is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s \\ -t \\ s \\ t \end{pmatrix} = s \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix},$$

which leads to the basis

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(c) $\dim \operatorname{Ker}(A) = 2$ and $\dim \operatorname{Im}(A) = \dim \operatorname{Col}(A) = 2$, as the respective bases have two elements each.

Ex 8.4 (Kernel and image)

(a) Let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be the linear transformation defined by $T(p) = p'$.
Find $\operatorname{Ker}(T)$ and $\operatorname{Im}(T)$, as well as bases for each of them.

(b) Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation defined by $T(p(t)) = \begin{pmatrix} p(0) \\ p'(0) \end{pmatrix}$,
where p' is the derivative of p. Find bases for $\operatorname{Ker}(T)$ and $\operatorname{Im}(T)$.

Solution:

(a) Exactly as done in a similar example in class, we can check that $\operatorname{Ker}(T) = \{a_0 : a_0 \in \mathbb{R}\} = \mathbb{P}_0$.
Moreover, we know that $\{1\}$ is a basis of $\mathbb{P}_0$ by Theorem 4.10.
So $\operatorname{Ker}(T) = \mathbb{P}_0$ and $\{1\}$ is a basis for it.

It is easy to observe that derivating drops the degree of a given polynomial by one.
Hence a good guess for $\operatorname{Im}(T)$ is: $\operatorname{Im}(T) = \mathbb{P}_2$.

Proof: Let $p \in \operatorname{Im}(T)$. This means that there exists $q \in \mathbb{P}_3$, so that $T(q) = p$.
We can write q as follows: $q = b_0 + b_1x + b_2x^2 + b_3x^3$.
Now, we can compute: $T(q) = q' = b_1 + 2b_2x + 3b_3x^2 = p$
So $p \in \mathbb{P}_2$ (its coefficients are $a_0 = b_1$, $a_1 = 2b_2$, $a_2 = 3b_3$).

For the other direction: Assume that $p \in \mathbb{P}_2$. So we can write $p = a_0 + a_1x + a_2x^2$.
We have to find a $q \in \mathbb{P}$ such that $T(q) = p$.
Define a candidate q by $q = a_0x + \frac{1}{2}a_1x^2 + \frac{1}{3}a_2x^3$.
We can now check (calculation) that indeed:
$T(q) = q' = a_0x + 2\frac{1}{2}a_1x^2 + 3\frac{1}{3}a_2x^3 = a_0 + a - 1x + a_2x^2 = p$.

(b) Claim: $\operatorname{Im}(T) = \mathbb{R}^2$.
Proof: Since obviously $\operatorname{Im}(T) \subset \mathbb{R}^2$, we are left to prove $\mathbb{R}^2 \subset \operatorname{Im}(T)$.
To this end, let $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$. We have to find $p \in \mathbb{P}_2$ so that $T(p) = \begin{pmatrix} a \\ b \end{pmatrix}$.
Define a candidate: $p = a + bx$ (of course, there are also other good candidates).
Then $p(0) = a_0$, $p' = b$ (for all x), and $p'(0) = b$.
So indeed $T(p) = \begin{pmatrix} a \\ b \end{pmatrix}$.

As a basis of $\mathrm{Im}(T)$ we can just take the standard basis of $\mathbb{R}^2$: $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$.

For $p = a_0 + a_1x + a_2x^2$, $T(a_0 + a_1x + a_2x^2) = \begin{pmatrix}a_0\\a_1\end{pmatrix}$.

So $T(p) = \begin{pmatrix}0\\0\end{pmatrix}$ if and only if $a_0 = a_1 = 0$ which means $p = a_2x^2$.
So $\mathrm{Ker}(T) = \{a_2x^2 : a_2 \in \mathbb{R}\} = \mathrm{Span}\{x^2\}$ and $\{x^2\}$ is a basis.

Ex 8.5 (A basis calculation)
Find a basis for the space spanned by the following vectors:

$$v_1 = \begin{pmatrix}1\\0\\-3\end{pmatrix}, \quad v_2 = \begin{pmatrix}0\\1\\2\end{pmatrix}, \quad v_3 = \begin{pmatrix}-1\\3\\9\end{pmatrix}, \quad v_4 = \begin{pmatrix}2\\1\\-6\end{pmatrix}.$$

Solution:
We compute the column echelon form of the matrix made with v_1, v_2, v_3 and v_4 as column vectors and find:

$$\begin{pmatrix}1 & 0 & -1 & 2\\0 & 1 & 3 & 1\\0 & 0 & 0 & -1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & -1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{pmatrix}.$$

and so the space spanned by the for vectors is simply $\mathbb{R}^3$ and a basis for this span would be

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

Alternative solution:
We can also compute the row echelon form for the above matrix. In particular, the pivot columns show that $\{v_1, v_2, v_4\}$ is a basis of the space $\mathrm{Span}(v_1, v_2, v_3, v_4)$.

Ex 8.6 (Getting acquainted with kernel and column space)
Let A be an $m \times n$ matrix, B an $n \times k$ matrix such that $\mathrm{Ker}(A) \cap \mathrm{Col}(B) = \{0\}$, and $\underline{b} = \{b_1, \cdots, b_k\}$ a basis of $\mathrm{Col}(B)$.
Show that $\mathcal{C} = \{Ab_1, \cdots, Ab_k\}$ is a basis of $\mathrm{Col}(AB)$.

Solution:
We first show that the set generates $\mathrm{Col}(AB)$. By definition, every element in $\mathrm{Col}(AB)$ is of the form $A(Bx)$ for some $x \in \mathbb{R}^q$. Since the vectors $b_1, \ldots, b_k$ generate $\mathrm{Col}(B)$, we can write $Bx = \sum_{i=1}^k \mu_i b_i$ for some $\mu_i \in \mathbb{R}$. Thus

$$ABx = \sum_{i=1}^{k} \mu_i Ab_i$$

and therefore $\{Ab_1, \ldots, Ab_k\}$ generates $\mathrm{Col}(AB)$.

Next we show that the set is linearly independent. Assume that

$$\lambda_1 Ab_1 + \ldots + \lambda_k Ab_k = 0.$$

Then $A(\lambda_1 b_1 + \ldots \lambda_k b_k) = 0$, so that $\lambda_1 b_1 + \ldots + \lambda_k b_k \in \mathrm{Ker}(A)$. But since the vectors $b_1, \ldots, b_k$ are a basis for $\mathrm{Col}(B)$, this linear combination also lies in $\mathrm{Col}(B)$ and from the assumption we infer that $\lambda_1 b_1 + \ldots + \lambda_k b_k = 0$. Since the vectors $b_1, \ldots, b_k$ are linearly independent, we deduce that $\lambda_1 = \ldots = \lambda_k = 0$, which proves the claim.

Ex 8.7 (Representing a vector in a different basis)
Let $\underline{b} = \{b_1, b_2, b_3\}$ be the basis of $\mathbb{R}^3$ with

$$b_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For the vector $u = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, determine $[u]_{\underline{b}}$.

Moreover, find the vector w such that $[w]_{\underline{b}} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$.

Solution:
For the first part, we want to solve $u = c_1 b_1 + c_2 b_2 + c_3 b_3$ for c_1, c_2, c_3, which is a system of three equations. So we use row reduction:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 2 & 1 & 1 & 1 \\ 0 & 3 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 3 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 4 & -9 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -\frac{9}{4} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{4} \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 1 & -\frac{9}{4} \end{array}\right) \quad \Rightarrow \quad [u]_{\underline{b}} = \frac{1}{4}\begin{pmatrix} 5 \\ 3 \\ -9 \end{pmatrix}.$$

The second part can be computed as follows:

$$w = 3 \cdot b_1 + 0 \cdot b_2 + (-1) b_3 = 3 \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}.$$

Ex 8.8 (More coordinate calculations)
We define:

$$\underline{b} = \left\{ \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ -7 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad [x]_{\underline{b}} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}. \quad \text{and} \quad y = \begin{pmatrix} 10 \\ -9 \\ 1 \end{pmatrix}$$

Find the vector x (*i.e.* its coordinates in the standard basis) and find $[y]_{\underline{b}}$.

Solution:
The coordinates $[x]_{\underline{b}}$ of x in the basis $\underline{b}$ are the coefficients of the development of x with the basis vectors b_1, b_2 and b_3. Thus, we have $x = 3b_1 - b_3 = \begin{pmatrix} -1 \\ -5 \\ 9 \end{pmatrix}$.

Here we are given $y = \begin{pmatrix} 10 \\ -9 \\ 1 \end{pmatrix}$ in the standard basis, and we have to find the coordinates of y in the $\underline{b}$ basis. It is the reverse calculation of the previous point. We are looking for a, b and c such that $ab_1 + bb_2 + cb_3 = y$. Solving the corresponding linear equation yields $[y]_{\underline{b}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Ex 8.9 (New coordinates for polynomials)

Determine $[t]_{\underline{b}}$ and $[1+t^2]_{\underline{b}}$ for the basis $\underline{b} = \{p_1, p_2, p_3\}$ of $\mathbb{P}_2$ where

$$p_1(t) = 1 + t + t^2, \quad p_2(t) = 2t - t^2, \quad p_3(t) = 2 + t - t^2.$$

Solution:

We do row reduction on the augmented matrix whose columns are the coefficients of the polynomials, and just for fun we'll do them at the same time (the fourth column is t and the fifth column $1+t^2$):

$$\left(\begin{array}{ccc|cc} 1 & 0 & 2 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 2 & 0 & 1 \\ 0 & 2 & -1 & 1 & -1 \\ 0 & -1 & -3 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 2 & -1 & 1 & -1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & -7 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{7} & \frac{1}{7} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 0 & \frac{2}{7} & \frac{5}{7} \\ 0 & 1 & 0 & \frac{3}{7} & -\frac{3}{7} \\ 0 & 0 & 1 & -\frac{1}{7} & \frac{1}{7} \end{array}\right)$$

$$\Longrightarrow \quad [t]_{\underline{b}} = \frac{1}{7}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \qquad [1+t^2]_{\underline{b}} = \frac{1}{7}\begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}.$$

Ex 8.10 (More polynomial calculations)

(a) Show that the set $F = \{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for $\mathbb{P}_2$.

(b) Find the coordinates vector of $T(t) = 1 + 4t + 7t^2$ in the basis F.

Solution:

(a) The standard basis of $\mathbb{P}_2$ is $\mathcal{E} = \{1, t, t^2\}$. In this basis, the coordinates of the elements of F are given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

We can now consider the matrix P whose columns are these vectors and do row reduction to show that it is equivalent to the identity matrix, which implies that F is indeed a basis.

(b) As the columns of P are the coordinates of the basis F in the standard basis, P is the matrix of the change of coordinates from basis F to the standard basis $\mathcal{E}$. We are looking for the coordinates of $f = 1 + 4t + 7t^2$ in this basis, so we have to solve the system $P_F[f]_F = [f]_{\mathcal{E}}$ where $[f]_{\mathcal{E}}$ is the vector containing the coordinates of f in the standard basis:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}.$$

The solution of this system is given by $[f]_F = \begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix}$.

Ex 8.11 (Dimension of the kernel) Let $A \in \mathbb{R}^{n\times n}$ and assume that the dimension of $\mathrm{Ker}(A) = 1$. Can $\dim \mathrm{Ker}(A^2)$ be equal to 0? Can it be equal to 1 or 2? Can it be larger than 2?

Tip: Start by trying to come up with a few simple examples of matrices A for which $\dim \mathrm{Ker}(A) = 1$ and check what $\dim \mathrm{Ker}(A^2)$ is.

(Beaware: the last question is more tricky than the others and is not a potential exam problem.)

Solution:
Once you start playing around with examples, you will very quickly find out that two cases are possible: $\dim \mathrm{Ker}(A^2) = 1$ and $\dim \mathrm{Ker}(A^2) = 2$. For example:

For $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we have $A^2 = A$ and $\dim \mathrm{Ker}(A) = 1$. This implies $\dim \mathrm{Ker}(A^2) = 1$.

For $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $\dim \mathrm{Ker}\, A = 1$ and $A^2 = 0$. So $\mathrm{Ker}(A^2) = \mathbb{R}^2$ and $\dim \mathrm{Ker}(A^2) = 2$.

You can easily generalize these examples to different sizes of matrices (try it!)

Playing around, you may have noticed that you cannot find examples for which $\dim \mathrm{Ker}(A^2)$ is 0 or greater > 2. In fact, both these cases are not possible. The fact that 0 is not possible is easy to prove:

$$\mathrm{Ker}(A) \subset \mathrm{Ker}(A^2), \text{ So } \dim \mathrm{Ker}(A^2) \geq \dim \mathrm{Ker}(A) = 1.$$

The fact that $\dim \mathrm{Ker}(A^2)$ cannot be greater than 2 is harder to prove.

The intuition (non-formally speaking!) is: an element of $\mathrm{Ker}(A^2)$ is either an element of $\mathrm{Ker}(A)$, or, it is mapped by A onto $\mathrm{Ker}(A)$. Each of these two cases provides one direction worth of vectors, that lie in $\mathrm{Ker}(A^2)$. In certain cases, these two directions coincide (see first example). So, in conclusion, we get at most two independent directions of vectors in $\mathrm{Ker}(A^2)$.

Here is a formal proof:
We prove by contradiction. Namely, we will assume $\dim \mathrm{Ker}(A^2) \geq 3$.

Since $\dim \mathrm{Ker}(A^2) \geq 3$, there exists $\{x_1, x_2, x_3\}$ linearly independent and $x_i \in \mathrm{Ker}(A^2)$ for all $i = 1, 2, 3$.

Since $\mathrm{Ker}\, A \subseteq \mathrm{Ker}(A^2)$ (since if $Ax = 0$ then $A^2x = 0$), we may assume that $x_1 \in \mathrm{Ker}\, A$.

Observe $Ax_i \in \mathrm{Ker}\, A$ for $i = 2, 3$ and so, as x_1 span $\mathrm{Ker}\, A$ (as it is one-dimensional), there exists $\lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$\lambda_2 x_1 = Ax_2, \quad \lambda_3 x_1 = Ax_3.$$

Furthermore, observe that λ_2, λ_3 are non-zero since if $\lambda_2 = 0$, then $x_2 \in \mathrm{Ker}\, A = \mathrm{span}\{x_1\}$ which implies that $\{x_1, x_2\}$ is linearly dependent (same argument holds for λ_3).

So dividing by λ_i, we have $x_1 = A(\lambda_i^{-1} x_i)$ for $i = 2, 3$.

Subtracting the two expressions, we get

$$0 = x_1 - x_1 = A(\lambda_2^{-1} x_2 - \lambda_3^{-1} x_3).$$

This implies $\lambda_2^{-1} x_2 - \lambda_3^{-1} x_3 \in \mathrm{Ker}\, A$.

As $\mathrm{Ker}\, A = \mathrm{span}\{x_1\}$, this implies there exists some $\lambda \in \mathbb{R}$ such that

$$\lambda_2^{-1} x_2 - \lambda_3^{-1} x_3 = \lambda x_1$$

which shows that $\{x_1, x_2, x_3\}$ is linearly dependent which is a contradiction.
Hence, $\dim \mathrm{Ker}(A^2) \leq 2$

Ex 8.12 (True/False questions)
Decide whether the following statements are always true or if they can be false.

(i) If $V = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$, then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis of V.

(ii) A spanning set of maximal size is a basis.

(iii) Suppose the matrix B is an echelon form of the matrix A. Then the pivot columns of B form a basis for $\text{Col}(A)$.

(iv) The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

(v) A linearly independent set in a subspace H is a basis for H.

(vi) If V is a vector space and $\underline{b}$ a basis with n elements, then $[x]_{\underline{b}}$ is a vector in $\mathbb{R}^n$.

(vii) If V is a vector space with a finite basis $\underline{b}$ and $P_{\underline{b}}$ is the change-of-coordinates matrix from $\underline{b}$ to the standard basis, then $[x]_{\underline{b}} = P_{\underline{b}} x$ for all $x \in V$.

Solution:

(i) **False.** To be a basis of V, the vectors would also have to be linearly independent. E.g., $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$ is not a basis of $\text{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right)$ because they are not independent.

(ii) **False.** A *linearly independent* set of maximal size must be a basis; a spanning set of *minimal* size must also be a basis. But a maximal spanning set would just be the whole vector space, which is obviously not a basis.

(iii) **False.** Columns *of A that correspond to* pivot columns of its reduced echelon form (i.e., columns $A_{i_1}, \dots, A_{i_k}$ of A such that B has a pivot in columns $i_1, \dots, i_k$) form a basis for $\text{Col}(A)$. (Those columns are (perhaps confusingly) called "pivot columns of A".) Two row equivalent matrices do not have to have the same column space, so a column of an echelon form need not even be in the column space of the original matrix

For instance, think of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$, whose reduced echelon form is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. The pivot columns of the reduced echelon form are $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$, but they do not form a basis for the column space of the original matrix, in fact the second vector is not even in that column space.

(iv) **True.** The columns of A are linearly independent because if a linear combination $x_1 A_1 + \ldots + x_n A_n$ of the columns of A is zero, then $Ax = 0$ and hence $x = A^{-1}Ax = A^{-1}0 = 0$. They also span $\mathbb{R}^n$ because $y = A(A^{-1}y)$ for all $y \in \mathbb{R}^n$.

(v) **False.** The set must also span H.

(vi) **True.** This is a consequence of the definition of the $\underline{b}$-coordinates.

(vii) **False.** By definition one has $P_{\underline{b}}[x]_{\underline{b}} = x$ for all $x \in V$.