MATH-111(en) FALL 2025

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SOLUTIONS for Homework 5

Do NOT use determinants to solve the problems on this Homework. Next weeks exercises will be full of problems about determinants. This week, I want you to practice other methods.

Ex 5.1 (Elementary matrix or not?)
Which of the following matrices are elementary matrices and what elementary operations do they represent?

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hint: Try either (or both) of the following strategies: (1) compare each matrix with the definitions of the three types of elementary matrices that we have seen in class; (2) Compute the matrix-matrix product of each of the matrix with another matrix.

Solution:
Since matrix-multiplication acts columnwise in the sense that $AB = (AB_1 \dots AB_n)$ it suffices to check the effect on general vectors. Let $x \in \mathbb{R}^3$. Then a direct computation shows that

$$Ax = \begin{pmatrix} x_1 \\ 2x_1 + x_2 \\ x_3 \end{pmatrix}$$

Thus A adds 2-times the first row to the second row and therefore it is an elementary matrix.
For $y \in \mathbb{R}^2$ we obtain

$$By = \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}.$$

Thus B exchanges the first and the second row and therefore it is an elementary matrix.
Again for $x \in \mathbb{R}^3$ we see that

$$Cx = \begin{pmatrix} x_1 + x_2 \\ x_2 \\ x_2 + x_3 \end{pmatrix}$$

and thus C adds the second row both to the first and the third row. In particular, it is not an elementary operation.
For the last matrix it suffices to note that $De_2 = 0$, so that D is not invertible and thus cannot be an elementary matrix (it corresponds to multiplying the second row by 0, which is not an elementary operation).

Ex 5.2 (Different methods for computing the inverse matrix)

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Compute A^{-1} using $\begin{cases} \text{(a) the formula for the inverse of a } 2\times 2 \text{ matrix,} \\ \text{(b) row reduction.} \end{cases}$

Solution:

(a)
$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{1\cdot 4 - 2\cdot 3}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

(b)
$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

So again $A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$.

Ex 5.3 (More inverse matrix calculations)

Compute the inverses of the following matrices:

$$(a)\ \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix} \qquad (c)\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Solution:

(a) By the formula for 2×2-matrices we have

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

(b) We use row reduction to find the inverse:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & -3 & -1 & -1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 8 & 2 & 3 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{8} & \frac{3}{8} & \frac{1}{8} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{8} & \frac{3}{8} & \frac{1}{8} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{8} & -\frac{5}{8} & \frac{1}{8} \\ 0 & 1 & 0 & \frac{2}{8} & -\frac{1}{8} & -\frac{3}{8} \\ 0 & 0 & 1 & \frac{2}{8} & \frac{3}{8} & \frac{1}{8} \end{array}\right)$$

$$\Longrightarrow \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix}^{-1} = \frac{1}{8}\begin{pmatrix} 2 & -5 & 1 \\ 2 & -1 & -3 \\ 2 & 3 & 1 \end{pmatrix}$$

(c) Again we use row reduction:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & -4 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 2 & -3 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 \end{array}\right) \Longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

Ex 5.4 (Determining invertibility)

Determine if the following matrices are invertible or not:

$$(a)\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad (b)\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & -1 \\ 3 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad (c)\begin{pmatrix} 0 & 2 & 3 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & -4 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

The matrix in (a) is not invertible because it is not a square matrix.
For the other two we will have to do row reduction, and since the question does not ask for the inverse, we can leave out the right side of the augmented matrix, and stop at the echelon form.

$$(b)\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & -1 \\ 3 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & -2 & -5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & -2 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -5 \end{pmatrix}$$

From this echelon form we can see that the matrix is invertible since it has 4 pivot elements.

$$(c)\begin{pmatrix} 0 & 2 & 3 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & -4 \\ 2 & 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & 2 & 2 & -4 \\ 2 & 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & 2 & 2 & -4 \\ 0 & 1 & 1 & -2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 3 & 0 \\ 0 & 2 & 2 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Here we see from the echelon form that the matrix is not invertible, because it just has 3 pivot elements.

Ex 5.5 (Inverting a linear transformation)

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the following linear transformation:

$$T(x) = \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 - 3x_3 \\ x_2 + x_3 \end{pmatrix}.$$

Prove that T is invertible and give a formula that defines the inverse transformation T^{-1} of T.

Solution:

We will invert the matrix corresponding to T, as usual with row reduction:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -4 & -3 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -4 & -3 & -2 & 1 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & -3 \\ 0 & 0 & 1 & -2 & 1 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 6 \\ 0 & 1 & 0 & 2 & -1 & -3 \\ 0 & 0 & 1 & -2 & 1 & 4 \end{array}\right)$$

As we were able to reach the identity matrix on the left-hand side by row transformations, it follows that T is invertible and that the left-hand side is the matrix of the inverse transformation T^{-1}. So,

$$T^{-1}(y) = \begin{pmatrix} -3 & 2 & 6 \\ 2 & -1 & -3 \\ -2 & 1 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -3y_1 + 2y_2 + 6y_3 \\ 2y_1 - y_2 - 3y_3 \\ -2y_1 + y_2 + 4y_3 \end{pmatrix}.$$

As a quick partial sanity check, let's compute the first entry of $S(T(x))$:

$$T^{-1}(T(x))_1 = -3(x_1+2x_2)+2(2x_1-3x_3)+6(x_2+x_3) = (-3+4)x_1+(-6+6)x_2+(-6+6)x_3 = x_1.$$

Ex 5.6 (Non-invertible matrices whose product is the identity matrix)

Find non-square matrices A and B such that $AB = I_n$ for some $n \in \mathbb{N}$ but A is not invertible.

Solution:
We can take for instance

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

If A were invertible, BA would also be an identity matrix, but we have

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \neq I_3.$$

Of course, we also know right away that A is not invertible from the fact that it is not a square matrix.

Ex 5.7 (Invertibility of factors of an invertible matrix)

Show that if A and B are $n \times n$ matrices and AB is invertible, then A and B are also invertible.

Solution:
If AB is invertible, then there is a C such that $ABC = I$. But then $AD = I$ for $D = BC$, which by the Invertible Matrix Theorem implies that A is invertible. In the same way we have $(CA)B = I$, which implies that B is invertible.

Ex 5.8 (Invertibility of elementary matrices)

a) Let $P_{ij} \in \mathbb{R}^{m \times m}$ be an elementary matrix that interchanges the ith and the jth row of a matrix. Verify that $P_{ij}^{-1} = P_{ij}$.

b) Let $D_i(\lambda) \in \mathbb{R}^{m \times m}$ be an elementary matrix that multiplies the ith row of a matrix by a scalar $\lambda \neq 0$. Verify that $D_i(\lambda)^{-1} = D_i(\lambda^{-1})$.

c) Let $L_{ij}(\lambda)$ be an elementary matrix that adds λ times the ith row to the jth row. Verify that $L_{ij}(\lambda)^{-1} = L_{ij}(-\lambda)$.

Hint: In all cases, calculate the claimed inverse times the matrix times I_m using the effect the corresponding matrices have.

Solution:

a) Since $P_{ij}I_m$ yields the identity matrix with the ith and the j row interchanged, we have $P_{ij}P_{ij} = P_{ij}P_{ij}I_m = I_m$ as the change is undone by the second multiplication with P_{ij}. From the Invertible Matrix Theorem we deduce that P_{ij} is invertible with $P_{ij}^{-1} = P_{ij}$.

b) Here the matrix $D_i(\lambda^{-1})$ acts by multiplying the ith row by the inverse of λ. Since $D_i(\lambda)$ multiplies the same row by λ, it holds that $D_i(\lambda^{-1})D_i(\lambda)I_m = I_m$ and again the claim follows from the Invertible Matrix Theorem.

c) The reason is similar. The matrix $L_{ij}(\lambda)$ only affects the jth row and by definition the jth row of $L_{ij}(\lambda)I_m$ is given by $e_j^T + \lambda e_i^T$ (here the transpose just means that we consider row vectors). By the same reasoning the jth row of $L_{ij}(-\lambda)L_{ij}(\lambda)I_m$ is $(e_j^T + \lambda e_i^T) - \lambda e_i^T = e_j$ since the ith row of $L_{ij}(\lambda)I_m$ is still e_i^T. Hence $L_{ij}(-\lambda)L_{ij}(\lambda)I_m = I_m$ and the claim follows from the Invertible Matrix Theorem.

Ex 5.9 (Multiple choice and True/False questions)

a) Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$. Then $(A^{-1})_{23}$ equals: (i) $-6/14$, (i) $-2/14$, (ii) $2/14$, (iv) $6/14$

b) Decide whether the following statements are always true or if they can be false.

(i) If A and B are invertible and of the same size, then $A + B$ is also invertible.

(ii) If $AB = AC$, then $B = C$.

(iii) If A is invertible and $AB = AC$, then $B = C$.

(iv) Every upper triangular matrix is in echelon form.

In the following, let A be a square matrix with n rows and n columns.

(v) If the equation $Ax = b$ has at least one solution for each $b \in \mathbb{R}^n$, then the solution is unique for each b.

(vi) If the columns of A are linearly independent, then they span $\mathbb{R}^n$.

(vii) If the columns of A span $\mathbb{R}^n$, then they are linearly independent.

(viii) If the columns of A are linearly independent, then its rows are also linearly independent.

(ix) If $n = 2$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ab - cd \neq 0$, then A is invertible.

(x) If A is invertible, then elementary row operations that reduce A to the identity I_n also reduce A^{-1} to I_n.

(xi) If there is an $n \times n$ matrix such that $AD = I$, then there is also an $n \times n$ matrix such that $CA = I$.

Solution:

a) **The answer is (iv).** Note that that in order to know $(A^{-1})_{23}$, it suffices to know the third column of A^{-1}, which is given by $A^{-1}e_3$. Hence we can save a little work by doing the calculation as follows (and once you understand this, you can leave out the dots).

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \cdot & \cdot & 0 \\ 3 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 2 & 2 & \cdot & \cdot & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \cdot & \cdot & 0 \\ 0 & 1 & -6 & \cdot & \cdot & 0 \\ 0 & 2 & 2 & \cdot & \cdot & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \cdot & \cdot & 0 \\ 0 & 1 & -6 & \cdot & \cdot & 0 \\ 0 & 0 & 14 & \cdot & \cdot & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \cdot & \cdot & 0 \\ 0 & 1 & -6 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \frac{1}{14} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \cdot & \cdot & -\frac{2}{14} \\ 0 & 1 & 0 & \cdot & \cdot & \frac{6}{14} \\ 0 & 0 & 1 & \cdot & \cdot & \frac{1}{14} \end{array}\right) \implies (A^{-1})_{23} = 6/14.$$

b) (i) **False.** Take $A = I_{2022}$ and $B = -I_{2022}$, then A and B are invertible, but $A + B = 0$ is not.

(ii) **False.** Here is a counterexample:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}.$$

(iii) **True.** Multiplying both sides of the equation $AB = AC$ by A^{-1} on the left gives

$$A^{-1}AB = A^{-1}AC \quad \Rightarrow \quad IB = IC \quad \Rightarrow \quad B = C.$$

(iv) **False.** $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ is an upper triangular matrix, but is not in echelon form.

(v) **True.** The if-statement is that (the linear transformation corresponding to) A is onto, the then-statement is that it is one-to-one. By the Invertible Matrix Theorem, these two statements are equivalent for square matrices.

(vi) **True.** These are also two equivalent statements by the Invertible Matrix Theorem. (One can also see it directly by noting that both are equivalent to having n pivots in the echelon form.)

(vii) **True.** As mentioned in the previous part, these statements are equivalent by the Invertible Matrix Theorem.

(viii) **True.** The rows of A are the columns of A^T. Because A is a square matrix and its columns are linearly independent, A is invertible, again by the Invertible Matrix Theorem. Then A^T is also invertible, so its columns, i.e. rows of A, are also linearly independent.

(ix) **False.** If $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, then $ab - cd = 1$, but A is not invertible. (Note that $ab - cd$ is not equal to $ad - bc$ in general, where the latter does "detect" invertibility.)

(x) **False.** In general, we need to apply the inverses of the row operations for A in the reverse order in order to reduce A^{-1} to I_n.

(xi) **True.** These two statements are equivalent by the Invertible Matrix Theorem.