MATH-111(en) Fall 2025
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SOLUTIONS for Homework 4

Ex 4.1 (Injective, surjective, bijective)

Each part in the below list defines a map $f : \mathbb{R}^n \to \mathbb{R}^m$. For each part, specify m and n; check whether they are linear; and check whether f is injective, surjective and/or bijective.

$$(a)\quad f\begin{pmatrix}x\\y\end{pmatrix} = 10x+y \qquad (b)\quad f\begin{pmatrix}x\\y\\z\end{pmatrix} = xy+z \qquad (c)\quad f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y+1\\2x\end{pmatrix}$$

$$(d)\quad f\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3x+y\\y\\3z+x\\y\end{pmatrix} \qquad (e)\quad f\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3x+y\\y\\3z+x\end{pmatrix}$$

Solution:
a) In this case $n=2$ and $m=1$. f is linear as

$$f\left(\lambda\begin{pmatrix}x_1\\y_2\end{pmatrix}+\begin{pmatrix}x_2\\y_2\end{pmatrix}\right) = f\begin{pmatrix}\lambda x_1+x_2\\\lambda y_1+y_2\end{pmatrix} = 10(\lambda x_1+x_2)+\lambda y_1+y_2 = \lambda f\begin{pmatrix}x_1\\y_2\end{pmatrix}+f\begin{pmatrix}x_2\\y_2\end{pmatrix}.$$

It is not injective as $\begin{pmatrix}1\\-10\end{pmatrix}$ is a non-trivial solution to $f(x)=0$ (alternatively, we can also conclude it is not injective as $n>m$ c.f. Corollary 1.15). It is surjective as for any $\lambda\in\mathbb{R}$, $f\begin{pmatrix}0\\\lambda\end{pmatrix}=\lambda$. It is not bijective as it is not injective.
b) $n=3, m=1$. f is not linear as

$$f\begin{pmatrix}1\\0\\0\end{pmatrix}+f\begin{pmatrix}0\\1\\0\end{pmatrix} = 0+0\neq 1 = f\begin{pmatrix}1\\1\\0\end{pmatrix} = f\left(\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}\right).$$

f is not injective as

$$f\begin{pmatrix}1\\0\\0\end{pmatrix} = f\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Note that we cannot use Corollary 1.15 here as f is not linear.

f is surjective since for any $\lambda\in\mathbb{R}$, $f\begin{pmatrix}0\\0\\\lambda\end{pmatrix}=\lambda$. Finally, it is not bijective as it is not injective.
c) $n=m=2$. Straightaway, we see that f is not linear as $f(0)\neq 0$. On the other hand, we observe that for $x\in\mathbb{R}^2$,

$$f(x) = Ax+v \text{ where } A=\begin{pmatrix}1&1\\2&0\end{pmatrix} \text{ and } v=\begin{pmatrix}1\\0\end{pmatrix}.$$

Moreover, by observing that the RREF of A has two pivots, it follows that A is bijective (i.e. injective and surjective. Thus, if $x, y \in \mathbb{R}^2$ is such that $f(x) = f(y)$, it follows that $Ax = Ay$ which by the injectivity of A implies that $x = y$. Consequently, f is injective.
On the other hand, for any $y \in \mathbb{R}^2$, as A is surjective, there exists a vector x such that $Ax = y - v$. Thus, $f(x) = Ax + v = y$ implying that f is surjective. Finally, as f is injective and surjective, we conclude that it is bijective.
d) $n = 3$ and $m = 4$. Since we know that matrix multiplication is linear, and

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

it follows that f is linear. Straightaway, we can conclude that f is not surjective as $m > n$ (alternatively, by performing row reduction as below, it is not surjective as the number of pivots $= 3 < 4 = m$). To check for injectivity, we do row reduction on A:

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, as A has $3 = n$ pivots, it follows that A and consequently f is injective. f is not bijective as it is not surjective.
e) $n = m = 3$. Again, we can write f as a matrix multiplication:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

So, f is linear. Performing row reduction on A, we find

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, as A has $3 = m = n$ pivots, A and consequently f is both injective and surjective and is thusly also bijective.

Ex 4.2 (Surjectivity depending on a parameter)

Find all $a \in \mathbb{R}$ so that the linear transformation with matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & a & 6 \\ 7 & 8 & 9 \end{pmatrix}$ is not surjective.

Solution:
We do row reduction:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & a & 6 \\ 7 & 8 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & a-8 & -6 \\ 0 & -6 & -12 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & a-8 & -6 \end{pmatrix}$$

If $a - 8 = -3$, the third row equals the second row, so we would get a row of zeros in the echelon form, which implies that it is not surjective. So for $a = 5$ the matrix is not surjective (and this is the only value of a for which this is the case).

Ex 4.3 (Composition of linear maps)
Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^p \to \mathbb{R}^n$ be linear maps. We define their composition $f \circ g : \mathbb{R}^p \to \mathbb{R}^m$ by $(f \circ g)(x) = f(g(x))$. Show that $f \circ g$ is a linear map, too.

Solution:
Let $x, y \in \mathbb{R}^p$ and $\lambda \in \mathbb{R}$. Then by definition of the composition we have

$$\begin{aligned}(f \circ g)(\lambda x + y) = f(g(\lambda x + y)) &\overset{\text{g linear}}{=} f(\lambda g(x) + g(y)) \\ &\overset{\text{f linear}}{=} \lambda f(g(x)) + f(g(y)) = \lambda (f \circ g)(x) + (f \circ g)(y).\end{aligned}$$

This shows the linearity of $f \circ g$.

Ex 4.4 (Some matrix products)
Let

$$A = \begin{pmatrix} 4 & -5 & 3 \\ 5 & 7 & -2 \\ -3 & 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & 0 & -1 \\ -1 & 5 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 5 \\ 4 & -3 \\ 1 & 0 \end{pmatrix}.$$

Compute AC, BC and CB.

Solution:

$$AC = \begin{pmatrix} -21 & 35 \\ 21 & 4 \\ 10 & -21 \end{pmatrix}, \quad BC = \begin{pmatrix} -8 & 35 \\ 23 & -20 \end{pmatrix} \quad \text{and} \quad CB = \begin{pmatrix} -12 & 25 & 11 \\ 31 & -15 & -10 \\ 7 & 0 & -1 \end{pmatrix}.$$

Ex 4.5 (When do these matrices commute?)
Consider the matrices

$$A = \begin{pmatrix} 3 & -4 \\ -5 & 1 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} 7 & 4 \\ 5 & k \end{pmatrix}.$$

For which values of k does the equality $AB = BA$ hold?

Solution:
We have

$$AB = \begin{pmatrix} 1 & 12 - 4k \\ -30 & k - 20 \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} 1 & -24 \\ 15 - 5k & k - 20 \end{pmatrix},$$

meaning that we must have $12 - 4k = -24$ and $15 - 5k = -30$ if $AB = BA$. The unique solution of this system of equations is $k = 9$, so $AB = BA$ if and only if $k = 9$.

Ex 4.6 (More matrix products)
Consider the matrices:

$$A = \begin{pmatrix} 7 & 0 \\ -1 & 5 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 \\ -4 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 7 \\ -3 \end{pmatrix}, \quad D = \begin{pmatrix} 8 & 2 \end{pmatrix}.$$

If they are defined, compute the matrices

$$AB, CA, CD, DC, DBC, BDB, A^T A \text{ and } AA^T.$$

For those that are not defined, explain why.

Solution:

$$AB = \begin{pmatrix} 7 & 28 \\ -21 & -4 \\ -9 & -4 \end{pmatrix}.$$

CA is not defined because we cannot multiply a 2×1 matrix by a 3×2 matrix.

$$CD = \begin{pmatrix} 56 & 14 \\ -24 & -6 \end{pmatrix}, \qquad DC = [50], \qquad DBC = [-96].$$

BD is not defined because we cannot multiply a 2×2 matrix by a 1×2 matrix; thus BDB is not defined either.

$$A^T A = \begin{pmatrix} 51 & -7 \\ -7 & 29 \end{pmatrix}, \qquad AA^T = \begin{pmatrix} 49 & -7 & -7 \\ -7 & 26 & 11 \\ -7 & 11 & 5 \end{pmatrix}.$$

Ex 4.7 (Multiplication by diagonal matrices)
Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

1. Compute AD and DA and explain how the rows and columns of A change when one multiplies A by D from the right and from the left.
2. Find all the diagonal matrices M of dimension 3×3 such that $AM = MA$.

Solution:

1. Multiplication on the right of A by D acts on the columns of A and multiplies them by 2, 3 and 4 respectively. Multiplication on the left of A by D acts on the rows of A and multiplies them by 2, 3 and 4 respectively.

$$AD = \begin{pmatrix} 2 & 3 & 4 \\ 2 & 6 & 12 \\ 2 & 12 & 20 \end{pmatrix} \quad \text{and} \quad DA = \begin{pmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 4 & 16 & 20 \end{pmatrix}.$$

2. Let M be and arbitrary diagonal matrix

$$M = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Then

$$AM = \begin{pmatrix} a & b & c \\ a & 2b & 3c \\ a & 4b & 5c \end{pmatrix}, \quad \text{and} \quad MA = \begin{pmatrix} a & a & a \\ b & 2b & 3b \\ c & 4c & 5c \end{pmatrix}.$$

In order to have $AM = MA$, it must be that $a = b$, $a = c$, and $3b = 3c$. Thus, any diagonal matrix M which commutes with A, i.e. $AM = MA$, can be written as $M = \lambda I_3$, $\lambda \in \mathbb{R}$.

Ex 4.8 (Inner and outer products)
We may consider any vector of $\mathbb{R}^n$ as an $n \times 1$ matrix. Let u and v in $\mathbb{R}^3$ be given as

$$u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

1. Write u^T and v^T.

We call $u^T\ v$ the scalar product (or dot product or inner product) of the vectors u and v.

2. What is the dimension of the products $u^T\ v$ and $v^T\ u$?
3. Are these two products equal for all possible values of a, b and c? Why?

The product $u\ v^T$ is called the outer product.

4. What are the dimensions of the products $u\ v^T$ and $v\ u^T$?
5. Are these two products equal for all possible values of a, b and c? Why?

Solution:

1. $u^T = (a\ b\ c)$ and $v^T = (1\ 2\ 3)$.
2. The matrix u^T is a 1×3 matrix and the matrix v is a 3×1 matrix. Thus the dimension of the product $u^T\ v$ is 1×1. Similarly, the dimension of the product $v^T\ u$ is 1×1 as well.
3. These products are the transpose of one another and are equal since they are of dimension 1.
4. The matrix u is a 3×1 matrix and the matrix v^T is a 1×3 matrix. Thus the dimension of the product $u\ v^T$ is 3×3. Similarly, the dimension of the product $v\ u^T$ is 3×3 as well.
5. No. The equality $v\ u^T = u\ v^T$ implies that $c = 3a$, so it cannot hold if we choose (e.g.) $(a, b, c) = (1, 0, 0)$. (It is again true that the two products are the transpose of one another, but a 3×3 is not equal to its transpose in general.)

Ex 4.9 (Upper triangular matrices)

1. Compute $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for $a, b \in \mathbb{R}$.
2. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and compute the following matrices: $A^8, A^T A, AA^T$.
3. Find a matrix B such that $\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \cdot B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
4. Prove the following statement: the transpose of an upper triangular matrix is always lower triangular.

5. Use the statement proven in the previous bullet point to argue that the transpose of a diagonal matrix is always diagonal.

Solution:

1.
$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}$$

2. Using (a), we see that $A^2 = A \cdot A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $A^4 = A^2 \cdot A^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ and hence
$$A^8 = A^4 \cdot A^4 = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}.$$
Moreover,
$$A^T A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad AA^T \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

3. Again, using (a) with $a = 5$ and $b = -5$ we see that $B = \begin{pmatrix} 1 & -5 \\ 0 & 1 \end{pmatrix}$ works.

4. Let $A \in \mathbb{R}^{n \times n}$ be an upper triangular matrix. This means that $a_{ij} = 0$ if $i > j$ (i.e if the first coefficient is higher than the second). Hence for the entries of A^T, we have: $(A^T)_{ij} = a_{ji} = 0$ if $i < j$. This means A^T is lower triangular.

5. Since $(A^T)^T = A$, we can deduce from the previous statement that for all lower triangular matrices A, the transpose is upper triangular. Now let A be diagonal ($a_{ij} = 0$ if $i \neq j$). This means A is both upper triangular and lower triangular. So its transpose A^T is also both upper triangular and lower triangular, hence diagonal.

Ex 4.10 (A matrix equation)
Find a solution X for the matrix equation
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{pmatrix}.$$

Solution:
As in the lecture we solve three systems at once, finding the columns of A. The augmented matrix reads :

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 3 & 5 \\ 0 & 1 & 1 & 2 & 4 & 6 \\ 1 & 0 & 1 & 3 & 5 & 7 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 3 & 5 \\ 0 & 1 & 1 & 2 & 4 & 6 \\ 0 & -1 & 1 & 2 & 2 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 3 & 5 \\ 0 & 1 & 1 & 2 & 4 & 6 \\ 0 & 0 & 2 & 4 & 6 & 8 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 3 & 5 \\ 0 & 1 & 1 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 3 & 5 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right)$$

The first system yields $X_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, the second one $X_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and the third one $X_3 = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$.

So $X = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 3 & 4 \end{pmatrix}$.

Ex 4.11 (On invertibility of matrices)
Let $A \in \mathbb{R}^{n \times n}$. We call A invertible if there exists $B \in \mathbb{R}^{n \times n}$ such that $AB = BA = I_n$.

a) Assume that A is invertible. Show that then $Ax = 0$ has only the trivial solution $x = 0$ and that $\text{span}(A_1, \ldots, A_n) = \mathbb{R}^n$, where $A_1, \ldots, A_n$ are the columns of A.

b) Assume that $n = 2$ and write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that A is invertible if and only if $ad - bc \neq 0$.
Hint: If $ad - bc = 0$, consider first the case $a = b = 0$ and otherwise the vector $x = \begin{pmatrix} -b \\ a \end{pmatrix}$. If $ad - bc \neq 0$, try the matrix $C = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Solution:
a) Assume that there exists $x \neq 0$ such that $Ax = 0$. Since A is invertible, there exists $B \in \mathbb{R}^{n \times n}$ such that $BA = I_n$. Hence $x = I_n x = (BA)x = B(Ax) = B0 = 0$, which gives a contradiction. Since also $AB = I_n$, for all $b \in \mathbb{R}^n$ we have $b = A(Bb)$, so that we can solve $Ax = b$ for every $b \in \mathbb{R}^n$. As shown in the course, this implies that $\text{span}(A_1, \ldots, A_n) = \mathbb{R}^n$.

b) If $ad - bc = 0$ and $a = b = 0$, then we have $A = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ and thus the reduced echelon form of A has at most one pivot element. In particular, $\text{span}(A_1, A_2) \neq \mathbb{R}^2$ and therefore A cannot be invertible due to a). If not both a and b equal zero, then the vector $\begin{pmatrix} -b \\ a \end{pmatrix}$ is non-zero and a direct computation yields

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} -ab + ba \\ -bc + da \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence according to a) cannot be invertible.

Now assume that $ad - bc \neq 0$. Then a direct computation yields

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus A is invertible.

Ex 4.12 (Multiple choice and True/False questions)

a) What x satisfies the following equation?

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 7 & -3 & -3 \\ -1 & x & 0 \\ -1 & 0 & x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(A) $x = -1$

(B) $x = 1$
(C) There is no such x.
(D) There are multiple such x's.

b) Decide whether the following statements are always true or if they can be false.

i) Let $A \in \mathbb{R}^{n\times n}$ for some $n \in \mathbb{N}$. Then $(A^2)^T = (A^T)^2$.
ii) Let $A \in \mathbb{R}^{n\times n}$ for some $n \in \mathbb{N}$. Then $A^T A = AA^T$.
iii) Let $A, B \in \mathbb{R}^{n\times n}$ for some $n \in \mathbb{N}$. Then $(AB)^2 = A^2B^2$.
iv) The transpose of a sum of matrices equals the sum of their transposes.
v) The transpose of a product of matrices equals the product of their transposes in the same order.

Solution:

a) **The answer is (B).** It is easy to check that $x = 1$ satisfies the equation. To see that there is no other solution, notice that the product of the second row of the first matrix and the second column of the second matrix is 1. This gives the equation $-3 + 4x = 1$, which has no other solution than $x = 1$.

b) i) **True.** We know $(AB)^T = B^T A^T$. Substituting $B = A$, we get $(A^2)^T = (A^T)^2$.

ii) **False.** For example, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a counterexample.

iii) **False.** Take, for example, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = A^T$. Then

$$(AB)^2 = (AA^T)^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix},$$

but

$$A^2B^2 = A^2(A^2)^T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

iv) **True.** Denoting the entry of a matrix C at the i-th row and the j-th column by $C_{i,j}$, we have

$$((A + B)^T)_{i,j} = (A + B)_{j,i} = A_{j,i} + B_{j,i} = (A^T)_{i,j} + (B^T)_{i,j} = (A^T + B^T)_{i,j}.$$

v) **False.** $(AB)^T = B^T A^T$, so the order is *reversed*.