MATH-111(en) FALL 2025
Linear Algebra

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SOLUTIONS for Homework 3

Ex 3.0 (Delay)
Finish Exercise 2.11 from last week if you have not done so yet.

Solution : See solution from previous sheet.

Ex 3.1 (Examples of linear (in)dependence)
Part 1 : Are the following sets of vectors linearly independent or linearly dependent ?

$$\text{(a)} \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix} \qquad \text{(b)} \begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\4\\1\end{pmatrix} \qquad \text{(c)} \begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

Part 2 : Are the following sets of vectors linearly independent or linearly dependent ? For this part, use Theorems 1.7, 1.8, and/or 1.9 to answer.

$$\text{(a)} \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}, \begin{pmatrix}10\\11\\12\end{pmatrix} \qquad \text{(b)} \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\4\\1\end{pmatrix} \qquad \text{(c)} \begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}1\\4\\1\end{pmatrix}$$

Part 3 : Consider the following set of vectors :

$$v_1 = \begin{pmatrix}2\\4\\2\end{pmatrix}, v_2 = \begin{pmatrix}1\\2\\1\end{pmatrix}, v_3 = \begin{pmatrix}1\\3\\1\end{pmatrix}$$

First, show that they are linearly dependent. Next, show that v_3 cannot be written as a linear combination of v_1 and v_2. Finally, explain how this does not contradict Theorem 1.7.

Solution :

Part 1 :
a) We solve the corresponding system and check if 0 is the only solution. To this end, we form the augmented matrix $(A|0)$, where A has as columns the vectors to check for linear dependence.

$$\left(\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & -3 & 0 \\ 0 & -6 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

From this echelon form we see that there are no free variables. Since 0 is always a solution of an homogeneous equation, Theorem 1.2. ii) implies that the solution is unique. That means the vectors are linearly independent.
Because there are only 2 vectors, there is an another (easier ?) way : We only have to check if one is a multiple of the other. That is clearly not the case here.

b) We use the same method as for a).

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

Again, there is a unique solution to this system, which can only be the zero vector, so the vectors are linearly independent.

$$\text{c)} \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

In this case there are infinitely many solutions (the third variable is free) , which means there must be solutions other than the zero vector, so these vectors are linearly dependent.

Part 2 :

a) There are two approaches to this question :
One method is to apply the first part of Theorem 1.8. Indeed, as there are more vectors than dimensions, Theorem 1.8 immediately tells us that the vectors are linearly dependent.
Another approach is to realize

$$-\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}.$$

Thus, as we have written one of the vectors as a linear combination of the others, the vectors are linearly dependent by Theorem 1.7.

b) This set of vectors contains the zero vector, hence it is linearly dependent by Theorem 1.8.

c) The vectors here are a subset of the vectors in Part 1(b). Since the vectors in Part 1(b) are independent, by Theorem 1.9 it follows that the vectors here are independent as well.

Part 3 :

The first part can either be solved the same way as Part 1c, or, you may notice that $v_1 = 2v_2$ which is also $v_1 = 2v_2 + 0v_3$, so by Theorem 1.7, the vectors v_1, v_2, v_3 are linearly dependent.
Assume v_3 can be written that as a linear combination of v_1 and v_2. This means that there exist scalar λ_1 and λ_2 such that $v_3 = \lambda_1 v_1 + \lambda_2 v_2$. Writing this out in numbers yields the system

$$\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \text{ which has augmented matrix } \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 4 & 2 & 3 \\ 2 & 1 & 1 \end{array}\right)$$

the usual arguments show that the system has no solution. In conclusion, v_3 cannot be written as a linear combination of v_1 and v_2.
Theorem 1.7 states that if the vectors v_1, v_2, v_3 are dependent, then *one of the* v_i can be written in terms of the other two. The theorem does not say *each of the* v_i. In the case here, the v_i can be chosen to be v_1 (as shown at the beginning of Part3) or v_2 (analogously) but not v_3 (as we have just proven).

Ex 3.2 (Proof of Corollary 1.10)

Use results from class to prove the following statements : For a matrix $A \in \mathbb{R}^{m \times n}$ with columns $A_1, \ldots, A_n \in \mathbb{R}^m$, the following hold

a) for all $b \in \mathbb{R}^m$ the equation $Ax = b$ has at most one solution if and only if the columns of A are linearly independent ;
b) for all $b \in \mathbb{R}^m$ the equation $Ax = b$ has at least one solution if and only if $\text{span}(A_1, \ldots, A_n) = \mathbb{R}^m$;
c) for all $b \in \mathbb{R}^m$ the equation $Ax = b$ has exactly one solution if and only if $\text{span}(A_1, \ldots, A_n) = \mathbb{R}^m$ and $\{A_1, \ldots, A_n\}$ are linearly independent.

Solution :
a) Suppose that the equation $Ax = b$ has at most one solution for all $b \in \mathbb{R}^m$, then we consider in particular $b = 0$, for which $x = 0$ is a solution. The equation $Ax = 0$ is given by

$$x_1 A_1 + \ldots + x_n A_n = 0,$$

so uniqueness of the solution $x = 0$ implies by definition that $A_1, \ldots, A_n$ are linearly independent.
If on the other hand the columns $A_1, \ldots, A_n$ are linearly independent, then the equation $Ax = 0$ has only the solution $x = 0$. Now consider $b \in \mathbb{R}^m$. If the equation $Ax = b$ has no solution, there is nothing to prove. If there exists a solution s_b, then by Theorem 1.6 it is the only solution as the solution set is given by $S = \{s_b + 0\} = \{s_b\}$.

b) This statement is the equivalence of i) and ii) of Theorem 1.4.

c) This is the combination of a) and b) since 'exactly one solution' is the logical combination of 'at most one solution' and 'at least one solution'.

Ex 3.3 (An equivalent definition of linear maps)
Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Show that f is linear if and only if

$$f(\lambda x + y) = \lambda f(x) + f(y) \qquad \text{for all } x, y \in \mathbb{R}^n,\ \lambda \in \mathbb{R}.$$

Hint: For one implication, check the lecture on Tuesday for a more general argument. For the other implication, insert special values of λ or y.

Solution :
Assume that f is linear and let $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then using the two properties of linear functions we have that

$$f(\lambda x + y) = f(\lambda x) + f(y) = \lambda f(x) + f(y).$$

Now assume that f satisfies the above equality. Taking $\lambda = 1 \in \mathbb{R}, x = y = 0 \in \mathbb{R}^n$ provides that

$$f(0) = f(1 \cdot 0 + 0) = 1 \cdot f(0) + f(0) = 2f(0)$$

which implies $f(0) = 0$. Now, selecting $y = 0 \in \mathbb{R}^n$ we have that

$$f(\lambda x) = f(\lambda x + 0) = \lambda f(x) + f(0) = \lambda f(x) + 0 = \lambda f(x),$$

as we had shown $f(0) = 0$. To prove the other property, we set $\lambda = 1$ and obtain immediately that $f(x + y) = f(x) + f(y)$. Thus f is linear.

Ex 3.4 (The weekly linear system : matrix equations and linear (in)dependence)

For each of the matrix equations : (i) solve the equation. (ii) From the solution to the equation, deduce whether the columns of the coefficient matrices are linearly independent or linearly dependent.

$$\text{(a)} \begin{pmatrix} 2 & -5 & 8 \\ -2 & -7 & 1 \\ 4 & 2 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 1 & -3 & 7 \\ -2 & 1 & -4 \\ 1 & 2 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution :

$$\text{(a)} \begin{pmatrix} 2 & -5 & 8 \\ -2 & -7 & 1 \\ 4 & 2 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This is an echelon form. Since there are fewer pivots (2) than variables (3), the system has infinitely many solutions. Thus there are nontrivial linear combinations of the columns of the coefficient matrix that sum to zero, so the columns are linearly dependent. To find the solution, we compute the reduced echelon form and find $x_2 = 3/4x_3$ and $x_1 = -17/8x_3$.

$$\text{(b)} \begin{pmatrix} 1 & -3 & 7 \\ -2 & 1 & -4 \\ 1 & 2 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 7 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & 2 & 9 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 5 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 0 & 12 & 0 \end{array}\right)$$

This is an echelon form. The number of pivots equals the number of variables, so the system has a unique solution, which is the zero vector. Hence there is no nontrivial linear combination of the columns of the coefficient matrix that equals zero, so the columns are linearly independent.

Ex 3.5 (Linear (in)dependence depending on a parameter)
For which values of $a \in \mathbb{R}$ are the following vectors linearly dependent ?

$$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ -6 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ a \\ 2 \end{pmatrix}$$

Solution :
We use the same method as for Ex. 3.1.

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 3 & -6 & a & 0 \\ -2 & 3 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 0 & a-3 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & a-3 & 0 \end{array}\right)$$

It follows from this echelon form that for $a = 3$ the vectors are linearly dependent. Indeed, in that case the last row is zero, and we have 2 pivots and 3 variables, so there are nontrivial solutions.

On the other hand, if $a \neq 3$, there are 3 pivots and 3 variables, so the zero vector is the only solution, meaning that the vectors are linearly independent.

Ex 3.6 (Vectors in the image of a linear transformation)
Consider the linear transformation (function) $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1 - 2x_2 \\ -x_1 \\ x_1 - 2x_2 \end{pmatrix}.$$

1. Find $x \in \mathbb{R}^2$ such that $T(x) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Are there any more such vectors x ?
2. Is there an $x \in \mathbb{R}^2$ such that $T(x) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$?
3. Is there any vector b such that $T(x) = b$ has more than one solution ?

Solution :
Recall that for linear transformations an equation $T(x) = b$ corresponds to a linear system.

a) We can once again solve this with row reduction.

$$\left(\begin{array}{cc|c} 2 & -2 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 2 \\ -1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & -2 & 3 \\ 0 & 2 & -3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{array}\right)$$

Hence $x_1 = -1, x_2 = -3/2$ is the unique solution, which gives the vector $x = \begin{pmatrix} -1 \\ -3/2 \end{pmatrix}$.
The fact that this solution is unique means that there are no other such vectors.

b)

$$\left(\begin{array}{cc|c} 2 & -2 & 2 \\ -1 & 0 & 1 \\ 1 & -2 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 2 \\ -1 & 0 & 1 \\ 2 & -2 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & -2 & 3 \\ 0 & 2 & -2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{array}\right)$$

From the last row of the echelon form we see that there is no solution x.

c) No. We saw in a) that there is some $b \in \mathbb{R}^3$ such that $T(x) = b$ has a unique solution. It follows from Theorem 1.6 that the equation $T(x) = 0$ has only the solution $x = 0$ and again by the same theorem there can be no other $b \in \mathbb{R}^3$ such that $T(x) = b$ has more than one solution. Here a more direct argument : it is easy to see that in the row reduction in a), it does not matter what numbers are in the last column, we will always get two pivots in the first two columns. So for any b, there is either a unique solution to $T(x) = b$ (if the third entry in the third column of the echelon form is 0), or there is no solution (if that entry is not 0).

Ex 3.7 (Visualizing linear transformations)

For the linear transformations given by the following matrices, draw a picture to show how they transform the unit square $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$.

$$\text{a)}\ \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \qquad \text{b)}\ \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \qquad \text{c)}\ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

Solution :

We use the method presented in the lecture and compute the vectors $T(0, 0)$, $T(1, 0)$, $T(0, 1)$ and $T(1, 1)$, which form the corners of the 'parallelogram' that is the image of the unit square under the linear transformation T.

a) The image of the unit square is the parallelogram with corners $(0, 0), (1, 0), (3, 1), (4, 1)$.

b) The rectangle with corners $(0, 0), (-1, 0), (0, -2), (-1, -2)$.

c) The line segment between $(-1, 0)$ and $(0, 0)$.

Ex 3.8 (Representing linear transformations with matrices)

Find the matrices of the transformations T determined by the equations below.

1. $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2z - y \\ 3y - 2x \\ 4x - 3z \end{pmatrix}$.

2. $T(x_1, x_2, x_3, x_4) = 3x_1 + 4x_3 - 2x_4$.

3. $T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}, \quad T\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix}, \quad T\left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}$.

   Hint: Express the vectors e_1, e_2, e_3 as linear combination of the vectors for which you know the image and then use linearity to compute what you need.

Solution :

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, one has to compute $T(e_i)$ for all unit vectors e_i $(i = 1, \ldots, n)$. The resulting vectors $T(e_i) \in \mathbb{R}^m$ build the columns of the matrix.

1. $\begin{pmatrix} 0 & -1 & 2 \\ -2 & 3 & 0 \\ 4 & 0 & -3 \end{pmatrix}$

2. $\begin{pmatrix} 3 & 0 & 4 & -2 \end{pmatrix}$

3. The first equation already tells us the first column of the matrix. To find the third column we note that

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Using the additive property of linear transformations we obtain

$$T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = T\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right) - T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -9 \\ 1 \\ 1 \end{pmatrix}$$

Then, to find the second column, we observe (or solve a linear system to find) that

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

so that

$$T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \frac{1}{2}T\left(\begin{pmatrix}1\\2\\1\end{pmatrix}\right) - \frac{1}{2}T\left(\begin{pmatrix}1\\0\\1\end{pmatrix}\right)$$
$$= \frac{1}{2}\begin{pmatrix}-3\\3\\2\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-5\\3\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

Thus the matrix of T is

$$\begin{pmatrix}4 & 1 & -9\\ 2 & 0 & 1\\ -1 & 1 & 1\end{pmatrix}.$$

Ex 3.9 (Injectivity/Surjectivity of linear maps)
For each of the following matrices, determine if the corresponding linear transformation is injective and/or surjective.

$$A = \begin{pmatrix}-2 & 4\\ 5 & 7\\ 1 & 0\end{pmatrix}, \quad B = \begin{pmatrix}2 & 1 & 0\\ 0 & 2 & 3\end{pmatrix}, \quad C = \begin{pmatrix}0 & 2 & 1\\ 2 & 4 & -1\\ 1 & 0 & 2\\ 1 & 0 & 1\end{pmatrix}.$$

Solution :
We put each matrix into echelon form and then we deduce the answer.

$$A = \begin{pmatrix}-2 & 4\\ 5 & 7\\ 1 & 0\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0\\ 5 & 7\\ -2 & 4\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0\\ 0 & 7\\ 0 & 4\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 4\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}$$

So the linear transformation of A is injective, because the number of pivots is the number of variables (so for any b, $Ax = b$ cannot have more than one solution). But it is not surjective, because there is a row of zeroes (so for some b, $Ax = b$ has no solution) (we could have known this in advance, since the matrix has more rows than columns).

$$B = \begin{pmatrix}2 & 1 & 0\\ 0 & 2 & 3\end{pmatrix}$$

The matrix B is already in echelon form. The linear transformation of B is not injective, because the echelon form has fewer pivots than variables (we could have seen this in advance from the fact that the matrix has fewer rows than columns). It is surjective, because there is no row of zeroes.

$$C = \begin{pmatrix}0 & 2 & 1\\ 2 & 4 & -1\\ 1 & 0 & 2\\ 1 & 0 & 1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 1\\ 2 & 4 & -1\\ 1 & 0 & 2\\ 0 & 2 & 1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 1\\ 0 & 4 & -3\\ 0 & 0 & 1\\ 0 & 2 & 1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\\ 0 & 2 & 0\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$$

The linear transformation of C is injective because it has the same number of pivots and variables, but not surjective because it has a row of zeroes.

Ex 3.10 (Multiple choice and True/False questions)

a) Let

$$a_1 = \begin{pmatrix} 3 \\ -5 \\ 2 \\ 8 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ h-10 \\ 13+h \\ 10 \end{pmatrix}$$

where h is a real number. For what value of h is the vector b in the plane generated by a_1 and a_2?

$$(A) \quad h=-6 \qquad (B) \quad h=0 \qquad (C) \quad h=-3 \qquad (D) \quad h=10.$$

b) For an $m \times n$ matrix A whose columns generate $\mathbb{R}^m$ it is always true that...

(A) The equation system $Ax = 0$ has a non-trivial solution.
(B) The echelon form of A has a pivot in each row.
(C) There are vectors b of $\mathbb{R}^m$ for which the system of equations $Ax = b$ is not consistent.
(D) The echelon form of the A matrix has a pivot in each column.

c) Decide whether the following statements are always true or if they can be false.

(i) The columns of any 4×5 matrix are linearly dependent.
(ii) If x and y are linearly independent, and if $\{x, y, z\}$ is linearly dependent, then z is in $\text{Span}\{x, y\}$.
(iii) If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.
(iv) If x and y are linearly independent, and if z is in $\text{Span}\{x, y\}$, then $\{x, y, z\}$ is linearly dependent.
(v) A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect on the columns of the $n \times n$ identity matrix.

Solution :

(a) **(A).** To find the answer, note that the plane generated by two vectors is just an equivalent formulation of the span generated by these vectors. Hence we need to consider the linear system corresponding to the augmented matrix $(a_1 \, a_2 \,|\, b)$, which has a solution only for $h = -6$.

(b) **(B).** This was a result proven in the course.

(c) (i) **True.** Call the matrix A. As A has more columns than rows, any row echelon form of it will have a column without a pivot, which corresponds to a free variable. Hence $Ax = 0$ has a non-trivial solution, meaning that the columns of A are linearly dependent.

(ii) **True.** If $\lambda x + \mu y + \kappa z = 0$ with $(\lambda, \mu, \kappa) \neq (0, 0, 0)$, then $\kappa \neq 0$ because otherwise $(\lambda, \mu) \neq (0, 0)$, and the latter would imply that x and y are linearly dependent. Hence $z = -\frac{\lambda}{\kappa}x - \frac{\mu}{\kappa}y \in \text{Span}\{x, y\}$.

(iii) **False.** The set could for example contain vectors that are multiples of each other.

(iv) **True.** If $z = \lambda x + \mu y$, then $\lambda x + \mu y - z$ is a non-trivial linear combination of x, y and z yielding 0.

(v) **True.** Every vector in $\mathbb{R}^n$ is a linear combination of the columns of the $n \times n$ identity matrix (which are precisely the vectors $e_1, \ldots, e_n$) and T preserves linear combinations.