MATH-111(en) FALL 2025
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SOLUTIONS for Homework 1

Ex 1.1 (Solving linear systems I)

Solve the following systems of linear equations by finding the augmented matrix and using row elimination.

$$a)\ \begin{cases} x+y &= 5 \\ 2x-5y &= 4 \end{cases} \qquad b)\ \begin{cases} x+y+z &= 3 \\ x-y &= 0 \\ x &= -46 \end{cases} \qquad c)\ \begin{cases} 2x+3y+z &= 0 \\ x-y+z &= 1 \\ 3x+2y+2z &= 1 \end{cases}$$

Solution :

Solving a system means : either determine that the system has no solutions, or determine that it has a unique solution and find it, or determine that it has infinitely many solutions and find a parametric description of all of them.

a)

$$\begin{cases} x+y &= 5 \\ 2x-5y &= 4 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & -5 & 4 \end{array}\right) \xrightarrow{2)\to 2)-2\cdot 1)} \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & -7 & -6 \end{array}\right) \xrightarrow{2)\to 2)/(-7)}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 6/7 \end{array}\right) \xrightarrow{1)\to 1)-\cdot 2)} \left(\begin{array}{cc|c} 1 & 0 & 29/7 \\ 0 & 1 & 6/7 \end{array}\right) \longrightarrow x = \frac{29}{7}, \quad y = \frac{6}{7}$$

So the system a) has exactly one solution which is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 29/7 \\ 6/7 \end{pmatrix}$.

Hence the solution space is $\mathcal{S} = \left\{ \begin{pmatrix} 29/7 \\ 6/7 \end{pmatrix} \right\}$ (a set that contains only one element).

b) If there are two operations in one step, the upper one is performed first.

$$\begin{cases} x+y+z &= 3 \\ x-y &= 0 \\ x &= -46 \end{cases} \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -46 \end{array}\right) \xrightarrow[3)\to 3)-1)]{2)\to 2)-\cdot 1)} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & -1 & -3 \\ 0 & -1 & -1 & -49 \end{array}\right)$$

$$\xrightarrow[3\to 3)-2\cdot 2)]{2)\leftrightarrow 3)} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -1 & -49 \\ 0 & 0 & 1 & 95 \end{array}\right) \xrightarrow[1)\to 1)-3)]{2)\to 2)+3)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & -92 \\ 0 & -1 & 0 & 46 \\ 0 & 0 & 1 & 95 \end{array}\right) \xrightarrow{1)\to 1)+2)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -46 \\ 0 & -1 & 0 & 46 \\ 0 & 0 & 1 & 95 \end{array}\right)$$

$$\longrightarrow x = -46, \quad y = -46, \quad z = 95$$

So the system b) has exactly one solution which is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -46 \\ -46 \\ 95 \end{pmatrix}$. Hence the solution space is $\mathcal{S} = \left\{ \begin{pmatrix} -46 \\ -46 \\ 95 \end{pmatrix} \right\}$ (set that contains only one element).

Comment : Swapping the second and third in second step is not necessary, but avoids fractions since you don't need to divide by 2. That can save time in the exam. Also, in this case it would be easier to use the equations directly : $x = -46$, $y = x = -46$,

$z = 3 - x - y = 95$. But to do that within the systematic row reduction, you would have to swap columns, which is not recommended (you would have to keep track of which variable corresponds to which column). With row reduction you are on the safe side.

c)

$$\begin{cases} 2x + 3y + z &= 0 \\ x - y + z &= 1 \\ 3x + 2y + 2z &= 1 \end{cases} \longrightarrow \left(\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & -1 & 1 & 1 \\ 3 & 2 & 2 & 1 \end{array}\right) \xrightarrow{1)\leftrightarrow 2)} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 \\ 3 & 2 & 2 & 1 \end{array}\right)$$

$$\underset{3)-3\cdot 3)}{\xrightarrow{2)\to 2)-2\cdot 1)2)}} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 5 & -1 & -2 \\ 0 & 5 & -1 & -2 \end{array}\right) \underset{2)\to 2)/5}{\xrightarrow{3)\to 3)-2)}} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1/5 & -2/5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{1)\to 1)+2)} \left(\begin{array}{ccc|c} 1 & 0 & 4/5 & 3/5 \\ 0 & 1 & -1/5 & -2/5 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow z \text{ is free}, \quad x = -\frac{4}{5}z + \frac{3}{5}, \quad y = \frac{1}{5}z - \frac{2}{5}$$

So the system c) has infinitely many solutions, namely all the vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ for which x and y are given by z (free) by $x = -\frac{4}{5}z + \frac{3}{5}, \quad y = \frac{1}{5}z - \frac{2}{5}$. Hence the solution space is $\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = -\frac{4}{5}z + \frac{3}{5}, \quad y = \frac{1}{5}z - \frac{2}{5}, \quad z \in \mathbb{R} \right\}$ (contains infinitely many elements).

Ex 1.2 (Solving linear systems II)
Find the augmented matrix of the following linear systems and use it to solve them.

$$a) \begin{cases} w - x + z &= 1 \\ x - 2y - z &= 0 \\ w + 3y &= 2 \end{cases} \qquad b) \begin{cases} 2x - 5y + 4z &= 0 \\ x + y + z &= 0 \\ 4x - 3y + 6z &= 1 \end{cases} \qquad c) \begin{cases} x - 2y + 3z &= 1 \\ 2x - 4y + 6z &= 2 \\ -x + 2y - 3z &= -1 \end{cases}$$

Solution :
a) We use the variables w, x, y, z in alphabetic order to find the corresponding matrix. One can also choose another ordering (which exchanges the columns). One only has to remember which variable corresponds to which column at the very end.

$$\begin{cases} w - x + z &= 1 \\ x - 2y - z &= 0 \\ w + 3y &= 2 \end{cases} \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 & 0 \\ 1 & 0 & 3 & 0 & 2 \end{array}\right) \xrightarrow{3)\to 3)-1)} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 \end{array}\right)$$

$$\underset{3)\to 3)/5}{\xrightarrow{3)\to 3)-2)}} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1/5 \end{array}\right) \underset{1)\to 1)+2)}{\xrightarrow{2)\to 2)+2\cdot 3)}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7/5 \\ 0 & 1 & 0 & -1 & 2/5 \\ 0 & 0 & 1 & 0 & 1/5 \end{array}\right)$$

$$\longrightarrow z \text{ is free}, \quad y = \frac{1}{5} \quad x = \frac{2}{5} + z \quad w = \frac{7}{5},,$$

Hence the solutions of the system are the elements of the space :

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : y = \frac{1}{5}, \quad x = \frac{2}{5} + z, \quad w = \frac{7}{5}, \quad z \in \mathbb{R} \right\}$$

b) Here we have three variables x, y, z.

$$\begin{cases} 2x - 5y + 4z &= 0 \\ x + y + z &= 0 \\ 4x - 3y + 6z &= 1 \end{cases} \longrightarrow \left(\begin{array}{ccc|c} 2 & -5 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 4 & -3 & 6 & 1 \end{array}\right) \overset{1)\leftrightarrow 2)}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -5 & 4 & 0 \\ 4 & -3 & 6 & 1 \end{array}\right)$$

$$\underset{3)\to 3)-4\cdot 1)}{\overset{2)\to 2)-2\cdot 1)}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & -7 & 2 & 1 \end{array}\right) \overset{3)\to 3)-2)}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \text{no solution (3rd row : } 0 \cdot z = 1)$$

Hence the solution space is the empty set $\mathcal{S} = \{\}$.

c)

$$\begin{cases} x - 2y + 3z &= 1 \\ 2x - 4y + 6z &= 2 \\ -x + 2y - 3z &= -1 \end{cases} \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & -4 & 6 & 2 \\ -1 & 2 & -3 & -1 \end{array}\right)$$

$$\underset{3)\to 3)+1)}{\overset{2)\to 2)-2\cdot 1))}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow y \text{ and } z \text{ are free,} \quad x = 2y - 3z + 1$$

Hence the solutions of the system are the elements of the space :

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \quad x = 2y - 3z + 1, \quad y, z \in \mathbb{R} \right\}$$

Ex 1.3 (Solving linear systems III)
Find the augmented matrix of the following linear systems and use it to solve them.

$$a) \begin{cases} w - x + y - z &= 1 \\ w + z &= 2 \end{cases} \qquad b) \begin{cases} x + y &= 0 \\ 3x + 5y &= 2 \\ 2x + 4y &= 2 \end{cases}$$

Solution :
a) Here we have four variables.

$$\begin{cases} w - x + y - z &= 1 \\ w + z &= 2 \end{cases} \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 2 \end{array}\right) \overset{2)\to 2)-1))}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 & 1 \end{array}\right)$$

$$\overset{1)\to 1)+2))}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & -1 & 2 & 1 \end{array}\right) \longrightarrow y, z \text{ free,} \quad x = y - 2z + 1, \quad w = -z + 2,$$

Hence the solutions space is

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : x = y - 2z + 1, \quad w = -z + 2, \quad y, z \in \mathbb{R} \right\}$$

b) There are two variables.

$$\begin{cases} x + y &= 0 \\ 3x + 5y &= 2 \\ 2x + 4y &= 2 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 3 & 5 & 2 \\ 2 & 4 & 2 \end{array}\right) \underset{3)\to 3)-2\cdot 1)}{\overset{2)\to 2)-3\cdot 1)}{\longrightarrow}} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{array}\right) \underset{2)\to 2)/2}{\overset{3)\to 3)-2))}{\longrightarrow}} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{1)\to 1)-2)} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow x=-1, y=1$$

Hence the solutions space is

$$\mathcal{S} = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

Ex 1.4 (Linear systems with a parameter)
Determine for which $a \in \mathbb{R}$ the following system has no solution, a unique solution, or infinitely many solutions.

$$\begin{cases} x - 2y + 3z &= 2 \\ x + 3y - 2z &= 5 \\ 2x - y + az &= 1 \end{cases}$$

Solution :
We apply row elimination for the augmented matrix. The answer will be clear from the (reduced) echelon form.

$$\begin{cases} x - 2y + 3z &= 2 \\ x + 3y - 2z &= 5 \\ 2x - y + az &= 1 \end{cases} \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 1 & 3 & -2 & 5 \\ 2 & -1 & a & 1 \end{array}\right) \underset{3)\to 3)-2\cdot 1)}{\xrightarrow{2)\to 2)-1)}} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 5 & -5 & 3 \\ 0 & 3 & a-6 & -3 \end{array}\right)$$

$$\underset{3)\to 3)-3\cdot 2)}{\xrightarrow{2)\to 2)/5}} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -1 & 3/5 \\ 0 & 0 & a-3 & -24/5 \end{array}\right)$$

We have arrived at a echelon form. To continue, we need to distinguish between $a = 3$ and $a \neq 3$. If $a = 3$, then the last line is incompatible since it requires $0 = -24/5$ and thus there exists no solution (if the last row were compatible, we continued to solve the system in a separate calculation). If $a \neq 3$, then $a - 3 \neq 0$ and we can divide by $a - 3$, leading to

$$\underset{2)\to 2)+3)}{\xrightarrow{3)\to 3)/(a-3)}} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & 0 & 3/5 - \frac{24}{5a-15} \\ 0 & 0 & 1 & \frac{-24}{5a-15} \end{array}\right) \underset{1)\to 1)+2\cdot 2)}{\xrightarrow{1)\to 1)-3\cdot 3)}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 16/5 + \frac{24}{5a-15} \\ 0 & 1 & 0 & 3/5 - \frac{24}{5a-15} \\ 0 & 0 & 1 & \frac{-24}{5a-15} \end{array}\right)$$

$$\longrightarrow \begin{cases} a = 3: & \text{no solutions} \\ a \neq 3: & \text{unique solution}: x = \frac{16}{5} + \frac{24}{5a-15}, \quad y = \frac{3}{5} - \frac{24}{5a-15}, \quad z = \frac{-24}{5a-15} \end{cases}$$

Note that to answer the question, we did not really have to determine the solutions as done here. From the echelon form we could see that there is no solution only if $a - 3 = 0$, and that otherwise there is a unique solution.

Ex 1.5 (Solvability of parameter-dependent systems)
Determine the values of h for which the following matrices are the augmented matrices of a consistent linear system. (A linear system is called *consistent* if it has at least one solution. It is *inconsistent* if there exists no solution.)

$$\text{(a)} \begin{pmatrix} 1 & -3 & h \\ -2 & 6 & -5 \end{pmatrix}, \qquad \text{(b)} \begin{pmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{pmatrix}.$$

Solution :
a) We use row elimination, which does not change the set of solutions :

$$\begin{pmatrix} 1 & -3 & h \\ -2 & 6 & -5 \end{pmatrix} \xrightarrow{2)\to 2)+2\cdot 1)} \begin{pmatrix} 1 & -3 & h \\ 0 & 0 & 2h-5 \end{pmatrix}.$$

The linear system corresponding to the last matrix is :

$$\begin{aligned} x - 3y &= h \\ 0 &= 2h - 5; \end{aligned}$$

it is consistent if and only if $2h - 5 = 0$, that is if $h = 5/2$.

b) Again by row elimination we find that

$$\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix} \overset{2)\to 2)-3\cdot 1)}{\longrightarrow} \begin{bmatrix} 1 & h & 4 \\ 0 & 6-3h & -4 \end{bmatrix}.$$

The linear system corresponding to the last matrix is

$$\begin{aligned} x + hy &= 4 \\ (6-3h)y &= -4\,. \end{aligned}$$

It is consistent if and only if $6 - 3h \neq 0$, that is if $h \neq 2$.

Ex 1.6 (Linear combinations)

a) For the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 11 \\ 16 \\ 21 \end{pmatrix}$, find $l, m \in \mathbb{R}$ such that $\mathbf{c} = l\mathbf{a} + m\mathbf{b}$.

b) Find all $a \in \mathbb{R}$ such that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \mathrm{Span}\{\begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ 2 \end{pmatrix}\}$.

Solution : a) Writing the equations for each component of the vector, we obtain a linear system that we solve with row elimination :

$$\begin{cases} l + 4m &= 11 \\ 2l + 5m &= 16 \\ 3l + 6m &= 21 \end{cases} \longrightarrow \left(\begin{array}{cc|c} 1 & 4 & 11 \\ 2 & 5 & 16 \\ 3 & 6 & 21 \end{array}\right) \underset{3)\to 3)-3\cdot 1)}{\overset{2)\to 2)-2\cdot 1)}{\longrightarrow}} \left(\begin{array}{cc|c} 1 & 4 & 11 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{array}\right) \underset{2)\to 2)/(-3)}{\overset{3)\to 3)-2\cdot 2)}{\longrightarrow}} \left(\begin{array}{cc|c} 1 & 4 & 11 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

$$\overset{1)\to 1)-4\cdot 2)}{\longrightarrow} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow l = 3, m = 2$$

Note that the same technique works for $(n+1)$ vectors in $\mathbb{R}^m$, which yields a $m \times n$-matrix.

b) By the definition of span as the set of linear combinations, again we need to express $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ as a linear combination of the vectors $\begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}$ and $\begin{pmatrix} a \\ 1 \\ 2 \end{pmatrix}$, which gives rise to a linear system :

$$\left(\begin{array}{cc|c} 1 & a & 1 \\ 0 & 1 & 1 \\ a & 2 & 0 \end{array}\right) \overset{3)\to 3)-a\cdot 1)}{\longrightarrow} \left(\begin{array}{cc|c} 1 & a & 1 \\ 0 & 1 & 1 \\ 0 & 2-a^2 & -a \end{array}\right) \overset{3)\to 3)-(2-a^2)\cdot 2)}{\longrightarrow} \left(\begin{array}{cc|c} 1 & a & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -a-(2-a^2) \end{array}\right)$$

So there is a solution if and only if $a^2 - a - 2 = 0$, which according to the well-known formula for zeros of quadratic equations holds if and only if $a = 2$ or $a = -1$.

Ex 1.7 (Linearity of linear systems)

Let

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ \vdots & \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m \end{cases}$$

be a general linear system.

a) Show that if $(s_1, \ldots, s_n)$ is a solution for the right-hand side $(b_1, \ldots, b_m)$ and $\lambda \in \mathbb{R}$ is a real number, then $(\lambda s_1, \ldots, \lambda s_n)$ is a solution for the right-hand side $(\lambda b_1, \ldots, \lambda b_m)$.

b) Show that if $(s_1, \ldots, s_n)$ is a solution for the right-hand side $(b_1, \ldots, b_m)$ and $(t_1, \ldots, t_n)$ is a solution for the right-hand side $(c_1, \ldots, c_m)$, then $(s_1 + t_1, \ldots, s_n + t_n)$ is a solution for the right-hand side $(b_1 + c_1, \ldots, b_m + c_m)$.

Remark: i) From a mathematical point of view, these two properties are the definition of a linear system.
ii) This is the first exercise which requires an abstract proof. If you don't know what to do, ask an assistant during the exercise sessions.

Solution :
a) We verify the statement for each linear equation. Consider the i-th equation with $1 \leq i \leq m$, which reads

$$a_{i1}x_1 + \ldots a_{in}x_n = b_i.$$

By assumption, we know that the choice $x_1 = s_1, \ldots, x_n = s_n$ gives a solution. Hence, for any $\lambda \in \mathbb{R}$ the standard calculus rules for real numbers yield that

$$a_{i1}(\lambda s_1) + \ldots + a_{in}(\lambda s_n) = \lambda a_{i1}s_1 + \ldots + \lambda a_{in}s_n = \lambda \underbrace{(a_{i1}s_1 + \ldots + a_{in}s_n)}_{=b_i} = \lambda b_i.$$

Since this is true for any i, we proved the claim.

b) The argument is similar to a). We fix the ith equation and insert $x_1 = s_1 + t_1, \ldots, x_n = s_n + t_n$ to find that

$$\begin{aligned} a_{i1}(s_1 + t_1) + \ldots + a_{in}(s_n + t_n) &= a_{i1}s_1 + a_{i1}t_1 + \ldots + a_{in}s_n + a_{in}t_n \\ &= \underbrace{a_{i1}s_1 + \ldots + a_{in}s_n}_{=b_i} + \underbrace{a_{i1}t_1 + \ldots a_{in}t_n}_{=c_i} = b_i + c_i. \end{aligned}$$

In the second equality we used the commutativity of the addition to reorder the terms in a convenient way, collecting firs the ones containing s and then the ones containing t.

Ex 1.8 (True/False with justification)
Determine if the following statements are true or false. Justify your answer, either with an argument or a counterexample.

1. If $u \in \text{Span}\{v_1, \ldots, v_n\}$, then $v_1 \in \text{Span}\{u, v_2, \ldots, v_n\}$.
2. For $u, v \in \mathbb{R}^3$ with $u \neq v$, $\text{Span}\{u, v\}$ is always a plane.

Solution :
1. **False :** Take for example $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $u \in \text{Span}\{v_1, v_2\}$ because $u = v_2$, but $v_1 \notin \text{Span}\{u, v_2\} = \text{Span}\{v_2\}$, since this only consists of vectors of the form $\begin{pmatrix} 0 \\ a \end{pmatrix}$ with $a \in \mathbb{R}$.

2. **False :** Consider for instance the vectors $u = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Then $\text{Span}\{u, v\} = \text{Span}\{v\}$ is a line.

Ex 1.9 (Multiple choice and True/False questions)

a) Multiple choice : Let R be the reduced echelon form of the matrix

$$\begin{pmatrix} 1 & 2 & -1 & -1 \\ 3 & 1 & 2 & -2 \\ 2 & 3 & -1 & -3 \end{pmatrix}$$

and denote by r_{ij} its entry located on the row i and column j. Then

$$(A) \quad r_{13} = -2 \qquad (B) \quad r_{13} = -1 \qquad (C) \quad r_{13} = 0 \qquad (D) \quad r_{13} = 1.$$

b) Multiple Choice : For each of the following statements, decide whether it always must be true or if it can be false.

(i) If the augmented matrix of a linear system has a row of the form $\begin{pmatrix} 0 & 0 & 0 & 0 & 4 & 0 \end{pmatrix}$, then there exists no solution.

(ii) When a linear system has free variables, then there exist infinitely many solutions.

(iii) If $v_1, \ldots, v_k$ are vectors in $\mathbb{R}^n$, then $0 \in \text{Span}\{v_1, \ldots, v_k\}$, where $0 \in \mathbb{R}^n$.

(iv) A linear system with right-hand side $b_1 = \ldots = b_m = 0$ always has a solution.

c) True/False : Every matrix can be brought into echelon form and that echelon form is unique.

d) True/False : Every system of linear equations with more unknowns than equations must have : (i) no solutions (ii) infinitely many solutions (iii) both are possible.

Solution :

a) The correct answer is (D). Just compute the reduced echelon form (you can stop as soon as you have the coefficient r_{13}).

b) (i) **False :** Just take this 1×6-matrix. The solution just requires that $x_5 = 0$.

(ii) **False :** It can have no solution. Consider for instance the augmented matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second variable is free, but the last row is incompatible.

(iii) **True :** Just write $0 = 0 \cdot v_1 + \ldots 0 \cdot v_n$.

(iii) **True :** The choice $x_1 = \ldots = x_n = 0$ is a solution.

c) The first part of the statement is true : indeed we can bring every matrix into row echelon form. This follows for example from Theorem 1.1 in class : every matrix can be transformed into reduced row echelon form. This implies that it is also in row echelon form.

However, the second part of the statement is false as (unlike reduced row echelon form) the representation of a matrix in row echelon form is not unique. We can prove this by giving a counterexample :

<u>Proof :</u> We define matrices A and $\tilde{A}$ as follows :

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \tilde{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We observe that both matrices are in row echelon form. Moreover, matrix A can be transformed into matrix $\tilde{A}$ by an elementary operation (subtract third row from second). Hence the two matrices are row equivalent but not identical.

d) Every system of linear equations with more unknowns than equations must have : (i) no solutions (ii) infinitely many solutions **(iii) both are possible**.

For example, the system on the left has no solutions, and, the system on the right has infinitely many solutions.

$$\begin{cases} x+y+z &= 1 \\ x+y+z &= 2 \end{cases} \qquad \begin{cases} x+y+z &= 1 \\ x-y &= 0 \end{cases}$$