MATH-111(en) FALL 2025
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SOLUTIONS for Homework 14

Ex 14.1 (Orthogonal diagonalization)

Orthogonally diagonalize the matrices $A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -3 & 0 & 0 \\ -3 & 12 & 0 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & -3 & 12 \end{pmatrix}$

Solution: Solution for A: The characteristic polynomial of A is

$$\chi_A(\lambda) = 50 - 15\lambda + \lambda^2 = (\lambda - 10)(\lambda - 5).$$

Its roots and hence the eigenvalues of A are $\lambda_1 = 10$, $\lambda_2 = 5$.

By solving $(10I_2 - A)v = 0$, we find the eigenvector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ for $\lambda_1 = 10$;

by solving $(5I_2 - A)v = 0$, we find the eigenvector and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for $\lambda_2 = 5$.

As A is symmetric, and v_1,v_2 are eigenvectors for different eigenvalues, we know that they are already orthogonal. So we only need to normalize them:

$u_1 := \frac{1}{||v_1||} v_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $u_2 := \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Set $U = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ and hence $A = UDU^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \left(\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \right)$.

(Observe that by coincidence, U is symmetric in this exercise.)

Solution for B: The characteristic polynomial of B is

$$\chi_B(\lambda) = \lambda^4 - 32\lambda^3 + 334\lambda^2 - 1248\lambda + 1521 = (\lambda^2 - 16\lambda + 39)^2 = (-13 + \lambda)^2(-3 + \lambda)^2.$$

(It's ok if you used your calculator to find the roots for this one.) So B has two eigenvalues 13 and 3. Each has algebraic multiplicity 2, so we write $\lambda_1 = \lambda_2 = 13$, $\lambda_3 = \lambda_4 = 3$. Since B is symmetric, each eigenvalue must also have geometric multiplicity = 2, i.e., each eigenvalue must have two independent eigenvectors. Find them by solving the equation $(13I_4 - B)v = 0$ respectively $(3I_4 - B)v = 0$, or, by clever guessing. This yields eigenvectors $v_1 = (-1, 3, 0, 0)^T$ and $v_2 = (0, 0, -1, 3)^T$ for 13; and $v_3 = (0, 0, 3, 1)^T$ and $v_4 = (3, 1, 0, 0)^T$ for 3. (Depending on how you solved the system, your vectors might look slightly different.)

Since B is symmetric, we know: every eigenvector of 13 is orthogonal to every eigenvector of 3. Hence, we only need to check orthogonality within $\{v_1, v_2\}$ and within $\{v_3, v_4\}$. And in fact, we can easily compute that $v_1 \cdot v_2 = 0$ and $v_3 \cdot v_4 = 0$.

(If that were not the case (e.g. if you found different eigenvectors), the orthogonalize by using Gram Schmidt, see example (*) below.)

Hence $\{v_1, v_2, v_3, v_4\}$ is an orthogonal eigenbasis for B. Normalizing $u_i := \frac{1}{||v_i||} v_i$ yields the following orthonormal eigenbasis of B: $u_1 = (\frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}}, 0, 0)^T$, $u_2 = (0, 0, \frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}})^T$, $u_3 = (0, 0, \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}})^T$, $u_4 = (\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0, 0)^T$. Hence

$$U = \begin{pmatrix} \frac{-1}{\sqrt{10}} & 0 & 0 & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & 0 & 0 & \frac{1}{\sqrt{10}} \\ 0 & \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & 0 \\ 0 & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 13 & 0 & 0 & 0 \\ 0 & 13 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad A = UDU^T$$

(∗) *For example, the eigenvectors of* 13 *that you have found might be* $v_1 = (-1, 3, -1, 3)^T$ *and* $v_2 = (0, 0, -1, 3)^T$. *In this case, they are not orthogonal* ($v_1 \times v_2 \neq 0$). *Applying Gram-Schmidt to* $\{v_1, v_2\}$ *yields* $u_1 = v_1 = (-1, 3, -1, 3)^T$, $u_2 = (\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}, \frac{3}{2})$. *Then proceed normalizing them and continue as above.*
In particular, there are many different possibilities for U, *depending on which* v_1, v_2, v_3, v_4 *that you started with. But there is only one solution for* D *(up to reordering the diagonal elements). You can easily verify whether the* U *that you found is correct by just computing* UDU^T *and checking whether the columns of your* U *are orthonormal.*

Ex 14.2 (Orthogonal diagonalization with some help)
Consider

$$A = \begin{pmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{pmatrix}, \quad v_1 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

1. Check that v_1 and v_2 are eigenvectors of A.
2. Orthogonally diagonalize the matrix A. (*Hint*: Make use of the fact that you already know two eigenvectors instead of just using the standard recipe for orthogonal diagonalization!)

Solution:

1. We have $Av_1 = 10v_1$ and $Av_2 = v_2$, so v_1 and v_2 are indeed eigenvectors of A.
2. Since A is symmetric it is diagonalisable. Moreover there is an (orthogonal) matrix P such as $A = PDP^T$. Being eigenvectors for different eigenvalues of a symmetric matrix, we know without calculation that v_1 and v_2 are two orthogonal eigenvectors. The third eigenvector has to be orthogonal to the two first ones. As we are in $\mathbb{R}^3$, the space which is orthogonal to v_1 and v_2 has dimension 1. All we have to do is to find a vector $v_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such as $v_1 \cdot v_3 = 0$ et $v_2 \cdot v_3 = 0$. We then solve : $\begin{pmatrix} -2 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

   The system has one free variable so we can write the solution as : $v_3 = z \begin{pmatrix} 1/4 \\ -1/4 \\ 1 \end{pmatrix}$

   We then check that $Av_3 = \lambda_3 v_3$ with $\lambda_3 = 1$. Finally, we need to normalize the three eigenvectors to obtain

   $$Q = \begin{pmatrix} -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{pmatrix}, \qquad D = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

   which allow us to write $A = QDQ^T$.

Ex 14.3 (Computing an SVD)
Find a singular value decomposition of each of the following matrices:

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Solution:

Recall the recipe: that: A singular value decomposition of a rank r matrix is a product $A = U\,\Sigma\,V^T$, where matrix Σ is "diagonal" and contains the r singular values $s_1, s_2, \ldots, s_r$ of A. They are the square roots of the non-zero eigenvalues of A^TA and we list them in a decreasing order. The matrix V is made from the orthonormalized eigenvectors of the symmetric matrix $A^TA : V = [v_1 v_2 \ldots]$.
Finally the matrix $U = [\mathbf{u}_1\mathbf{u}_2 \ldots]$ is such that $\mathbf{u}_i = \frac{1}{s_i}Av_i$ for all $1 \le i \le r$, and the remaining vectors $u_{r+1}, ..., u_m$ (if necessary) can be found by completing the $u_1, \ldots, u_r$ to an orthonormal basis for $\mathbb{R}^m$.

SVD for A: We have $A^TA = \begin{pmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{pmatrix}$. The characteristic polynomial is given by

$$\det(\lambda I - A^TA) = \lambda^3 - 34\lambda^2 + 225\lambda.$$

We see directly that 0 is an eigenvalue. Diving by λ, we have a polynomial of degree two and the corresponding eigenvalues are 9 and 25. We arranged them in decreasing order: $\lambda_1 = 25, \lambda_2 = 9, \lambda_3 = 0$. Since there are three distinct eigenvalues, the eigenspaces are one-dimensional and orthogonal. We find them by the usual row reduction. After normalizing, they are as follows:

$$v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -\frac{1}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \\ -\frac{4}{\sqrt{18}} \end{pmatrix}, \quad v_3 = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

(Since in this problem, each eigenspace is one dimensional, your v_1, v_2, v_3 must be exactly the above ones up to reversing direction.)

Since λ_1 and λ_2 are > 0 but $\lambda_3 = 0$, we have that $r = 2$ and $\sigma_1 = 5$ and $\sigma_2 = 3$. This already gives us the matrix $\Sigma \in \mathbb{R}^{2\times 3}$.

To compute U, we only need to compute the vectors Av_1 and Av_2 (they are orthogonal by Lemma 7.3) and normalize. This gives u_1, u_2.

$$u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad u_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Since here $r = m = 2$, the $\{u_1, u_2\}$ is already a basis of $\mathbb{R}^m = \mathbb{R}^2$, so no basis expansion is required. $U = (u_1, u_2) \in \mathbb{R}^{2\times 2}$.

So in conclusion: $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{4}{3\sqrt{2}} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$.

<u>SVD for B:</u> B^TB has the eigenvalues 9 and 4. With the same strategy as for A we find that

$$B = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

<u>SVD for C:</u> C^TC has the eigenvalues 3 and 2, so

$$C = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here the last column of U is found as a vector of unit length that is orthogonal to the first two columns (solve the corresponding linear system, then normalize).

Ex 14.4 (SVD with higher geometric multiplicity)
Find a singular value decomposition of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution:
Firstly, compute

$$A^TA = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 \\ 0 & 0 & 9 & -12 & 0 \\ 0 & 0 & -12 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for which it has the characteristic polynomial

$$\chi_{A^TA}(\lambda) = (\lambda - 25)^2(\lambda - 1)\lambda^2.$$

Thus, A^TA has eigenvalues $25, 1$ and 0 of which the eigenvalues 25 and 0 has algebraic and geometric multiplicity of 2. Namely, if A has SVD $U\Sigma V^T$, then

$$\Sigma = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, the eigenvectors of A^TA are (depending on your method you might find different eigenvectors!):

- Eigenvalue 25: $\tilde{v}_1 = e_2$ and $\tilde{v}_2 = 2e_2 - 3e_3 + 4e_4$,
- Eigenvalue 1: $v_3 = e_1$,
- Eigenvalue 0: $v_4 = 4e_3 + 3e_4$ and $v_5 = e_5$.

Note that at this point, we cannot directly input the normalized version of these vectors into the matrix V since the vectors corresponding to the eigenvalue 25 are not yet orthogonal. Thus, to make them orthogonal, we apply the Gram-Schmidt procedure to $\tilde{v}_1$ and $\tilde{v}_2$ which gives us

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ and } v_2 = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -4 \\ 0 \end{pmatrix}.$$

Now, by normalizing these vectors and setting them into V, we have

$$V = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{5} & 0 & \frac{4}{5} & 0 \\ 0 & -\frac{4}{5} & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, to find U, we compute $\tilde{u}_i = Av_i$ for $i = 1, 2, 3$:

$$\tilde{u}_1 = \begin{pmatrix} 0 \\ 5 \\ 0 \\ 0 \end{pmatrix}, \tilde{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 25 \\ 0 \end{pmatrix}, \tilde{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Normalizing,

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, completing the ONB with $u_4 = e_4$,

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In conclusion, the following is an SVD for A

$$A = U\Sigma V^T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{5} & 0 & \frac{4}{5} & 0 \\ 0 & -\frac{4}{5} & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

Ex 14.5 (A proof using SVD)

We say that two matrices $A, B \in \mathbb{R}^{n\times n}$ are called *orthogonally similar* if there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that $A = QBQ^T$. Let $A \in \mathbb{R}^{n\times n}$. Show that A^TA and AA^T are orthogonally similar.

Hint: Use a SVD of A and that the product of orthogonal matrices is orthogonal.

You do not need to memorize the definition of *orthogonally similar* for the exam. This exercise is just for your to train your proof-writing within the topics of Section 7.

Solution:

Let us write $A = U\Sigma V^T$ in the form of a SVD. Since A is square, the matrix Σ is square, too and therefore a diagonal matrix. Hence $A^T = V\Sigma^T U^T = V\Sigma U^T$. Thus we can compute

$$A^T A = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T, \qquad AA^T = U\Sigma V^T V\Sigma U^T = U\Sigma^2 U^T.$$

This implies that

$$A^T A = V\Sigma^2 V^T = VU^T AA^T UV^T = VU^T AA^T (VU^T)^T,$$

so the claim follows upon setting $Q = VU^T$.

Ex 14.6 (Computing SVD from eigenvectors)

Let $A \in \mathbb{R}^{2\times 4}$, $w_1, w_2 \in \mathbb{R}^4$ be such that w_1, w_2 are eigenvectors of $A^T A$, and

$$w_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \; w_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \; Aw_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \; Aw_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Find matrices U, Σ and V such that A has singular value decomposition of the form

$$A = U\Sigma V^T.$$

Solution:

Sizes of the matrices: We observe that w_i for $i = 1, 2$ are vectors in $\mathbb{R}^4$ and the product Aw_i is well-defined. Hence, A must have four 4 columns.
Moreover, since Aw_i for $i = 1, 2$ are vectors in $\mathbb{R}^2$, we know that A has 2 rows. Therefore, $A \in \mathbb{R}_{2\times 4}$. This implies that $\Sigma \in \mathbb{R}_{2\times 4}$, $U \in \mathbb{R}_{2\times 2}$ and $V \in \mathbb{R}_{4\times 4}$.

We first compute $\Sigma \in \mathbb{R}_{2\times 4}$: We observe that $w_1 \cdot w_2 = 0$, and that $\|w_1\| = \sqrt{2}$, $\|w_2\| = \sqrt{3}$. Since $w_i \in \mathbb{R}^4$ for $i = 1, 2$ are not normalized eigenvectors of $A^T A$, first define

$$v_1 = \frac{w_1}{\|w_1\|} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} \text{ et } v_2 = \frac{w_2}{\|w_2\|} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{pmatrix}.$$

We conclude that $v_i \in \mathbb{R}^4$ for $i = 1, 2$ are normalized eigenvectors of $A^T A$. Moreover,

$$\|Av_1\| = \left\| \frac{Aw_1}{\|w_1\|} \right\| = \frac{1}{\sqrt{2}} \|Aw_1\| = \frac{\sqrt{5}}{\sqrt{2}},$$
$$\|Av_2\| = \left\| \frac{Aw_2}{\|w_2\|} \right\| = \frac{1}{\sqrt{3}} \|Aw_2\| = \frac{\sqrt{5}}{\sqrt{3}}.$$

In consequence, the singular values of A are, $\sigma_1 = \sqrt{5}/\sqrt{2}$ and $\sigma_2 = \sqrt{5}/\sqrt{3}$. We made sure that $\sigma_1 > \sigma_2$ (descending order), and hence

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{5}/\sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{5}/\sqrt{3} & 0 & 0 \end{pmatrix}.$$

Let us now compute $V \in \mathbb{R}^{4\times 4}$: The first two columns of V are the normalized versions of v_1 and v_2. The other two columns are the eigenvectors of $A^T A$ for the eigenvalue 0.

To find the other two columns v_3, v_4 of V, it suffices to expand $\{v_1, v_2\}$ to an orthonormal basis $\{v_1, v_2, v_3, v_4\}$ of $\mathbb{R}^4$.
Why is that a good strategy here? There are two ways to understand this:
1.) following our recipe from class, the missing two columns of V are the eigenvectors of $A^T A$ for the eigenvalue 0. They are orthogonal to v_1 and v_2 as $A^T A$ is symmetric. So if we find two orthogonal vectors that are orthogonal to v_1, v_2, then they must be eigenvectors for 0.
2.) Look at the product $U\Sigma V^T$. Since Σ has all zeros in the last two columns, the last two rows of V^T (which are the last two columns of V) will not matter in the product. So we can choose them to be whatever as long as V is an orthogonal matrix.

So let us now find v_3, v_4, so that $\{v_1, v_2, v_3, v_4\}$ is an orthonormal basis of $\mathbb{R}^4$: There are various ways to achieve this; see e.g. the example at the end of the lecture notes of Week 13. Depending on the approach that you choose, your solution for v_3, v_4 might look slightly different than the one presented here, which is totally fine.
The strategy we use here, is finding a basis for the orthogonal complement of $\operatorname{Span} v_2, v_2$ an orthonormalizing it. To find the orthogonal complement of $\operatorname{Span} v_2, v_2$, we compute the kernel of the matrix

$$(w_1\, w_2)^T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

Its reduced echelon form is:

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \overset{L_2 \leftarrow L_2 - L_1}{\longrightarrow} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix} \overset{L_2 \leftarrow \frac{1}{2} L_2}{\longrightarrow} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \end{pmatrix} \overset{L_1 \leftarrow L_1 + L_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \end{pmatrix}.$$

And we get that

$$\operatorname{Ker} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_1 = x_2 = -x_3/2 \right\}$$

$$= \left\{ x_3 \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} : x_3, x_4 \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

So, a basis of that kernel and hence of the orthogonal complement of $\operatorname{Span}\{v_1, v_2\}$ is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

We observe that this basis is already orthogonal. Hence, we only need to normalize and obtain

$$v_3 = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \\ 0 \end{pmatrix} \text{ et } v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence, $\{v_1, v_2, v_3, v_4\}$ is an orthonormal basis of $\mathbb{R}^4$. And we can define the orhogonal matrix

$$V = (v_1\, v_2\, v_3\, v_4) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, we compute the matrix $U \in \mathbb{R}^{2\times 2}$: We first observe that $r = 2$, since A has two singular values). So the first two columns of U are the normalized versions of Av_1 and Av_2. But U only has two columns, so we already have all of them.
So, let us normalize Av_1 and Av_2:

$$u_i = \frac{Av_i}{\|Av_i\|} = \frac{Av_i}{\sigma_i}$$

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{Aw_1}{\sigma_1\|w_1\|} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix},$$
$$u_2 = \frac{Av_2}{\sigma_2} = \frac{Aw_2}{\sigma_2\|w_2\|} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}.$$

Hence,

$$U = (u_1\, u_2) = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

Finally, we have:

$$A = U\Sigma V^T = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{5}{2}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{5}{3}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

Ex 14.7 (Calculating $\exp(tA)$ and solving ODEs)

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. (a) Compute $\exp(tA)$. (**Hint**: Diagonalize A.)

b) Solve the differential equation $x'(t) = A \cdot x(t)$ for each of the initial values:
(i) $x(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, (ii) $x(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

Solution:

a) The characteristic polynomial of A is

$$(t-2)^2 - 1 = t^2 - 4t + 3 = (t-1)(t-3),$$

so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. Solving $Ax - I = 0$ and $Ax - 3I = 0$, we see that the $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector with eigenvalue 1 and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 3. Hence A can be diagonalized as $A = PDP^{-1}$, where

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \text{ with } P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Hence, as seen in the lecture,

$$\exp(tA) = \exp(tPDP^{-1}) = P\exp(tD)P^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} e^t + e^{3t} & e^{3t} - e^t \\ e^{3t} - e^t & e^t + e^{3t} \end{pmatrix}.$$

b) Recall that every solution of $x'(t) = A \cdot x(t)$ satisfies $x(t) = \exp(tA) \cdot x(0)$, meaning that we have the following solutions.

i)
$$x(t) = \frac{1}{2}\begin{pmatrix} \exp(t)+\exp(3t) & \exp(3t)-\exp(t) \\ \exp(3t)-\exp(t) & \exp(t)+\exp(3t) \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\exp(t) \\ \exp(t) \end{pmatrix}.$$

ii)
$$x(t) = \frac{1}{2}\begin{pmatrix} \exp(t)+\exp(3t) & \exp(3t)-\exp(t) \\ \exp(3t)-\exp(t) & \exp(t)+\exp(3t) \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} \exp(t)+3\exp(3t) \\ 3\exp(3t)-\exp(t) \end{pmatrix}.$$

Ex 14.8 (Solving ODEs)
Solve the following system of differential equations :
$$\begin{cases} x_1'(t) &= 5x_1(t) - 4x_2(t) - 2x_3(t) \\ x_2'(t) &= -4x_1(t) + 5x_2(t) + 2x_3(t) \\ x_3'(t) &= -2x_1(t) + 2x_2(t) + 2x_3(t) \end{cases}$$
for the initial values $x_1(0) = 0$, $x_2(0) = 0$, $x_3(0) = 1$

Hint: transfer it into a suitable matrix form. Then before your start investing loads of time into computations, ask yourself whether the matrix looks familiar to you.

Solution: Transferring this system into matrix form yields
$$\begin{pmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix}$$

Notice that for this matrix we have already found a diagonlization in Exercise 14.2. (The fact that it is an *orthogonal* diagonalization instead of just any diagonalization does not make things better or worse here.) Namely, $A = QDQ^{-1}$ where Q and D are as in the solution of Exercise 14.3 and $Q^{-1} = Q^T$. Hence
$$\begin{aligned} \exp(tA) &= Q\exp(tD)Q^T \\ &= \begin{pmatrix} -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{pmatrix}\begin{pmatrix} e^{t10} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix}\begin{pmatrix} -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{pmatrix}^T \\ &= \begin{pmatrix} \frac{1}{9}(4e^{10t}+5e^t) & \frac{1}{9}(-4e^{10t}+4e^t) & \frac{2}{9}(-e^{10t}+e^t) \\ \frac{1}{9}(-4e^{10t}+4e^t) & \frac{1}{9}(4e^{10t}+5e^t) & \frac{1}{9}(2e^{10t}-2e^t) \\ \frac{1}{9}(-2e^{10^t}+2e^t) & \frac{1}{9}(2e^{10^t}-2e^t) & \frac{1}{9}(e^{10t}+8e^t) \end{pmatrix} \end{aligned}$$

and solutions for our system of linear equations are of the form
$$\exp(tA)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \text{for parameters } c_1, c_2, c_3 \in \mathbb{R}.$$

Solving
$$\exp(0 \cdot A)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

for c_1, c_2, c_3 yields $c_1 = 0, c_2 = 0, c_3 = 1$ In conclusion, the solution for Ex.14.7(a) is

$$\begin{aligned}
\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} &= \exp(tA) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{9}(4e^{10t} + 5e^t) & \frac{1}{9}(-4e^{10t} + 4e^t) & \frac{2}{9}(-e^{10t} + e^t) \\ \frac{1}{9}(-4e^{10t} + 4e^t) & \frac{1}{9}(4e^{10t} + 5e^t) & \frac{1}{9}(2e^{10t} - 2e^t) \\ \frac{1}{9}(-2e^{10t} + 2e^t) & \frac{1}{9}(2e^{10t} - 2e^t) & \frac{1}{9}(e^{10t} + 8e^t) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{2}{9}(-e^{10t} + e^t) \\ \frac{1}{9}(2e^{10t} - 2e^t) \\ \frac{1}{9}(e^{10t} + 8e^t) \end{pmatrix}.
\end{aligned}$$

Ex 14.9 (A higher order ODE)
Consider the following differential equation: $y'''(t) + 4y''(t) - 4y'(t) = 0$

(a) Transform this ODE of order n into a system of ODEs of order 1 and write it in matrix-vector-form.

(b) Compute $\exp(tA)$ for $t \in \mathbb{R}$.

(c) Using the method of matrix exponentials, compute a solution $y(t)$ for the differential equation for the initial values: $y''(0) = y'(0) = 0$ and $y(0) = 1$?

Solution: (Straightaway, we see that the constant solution $y(t) = 1$ satisfy the SDE. However, for practice, let us try the method involving matrix exponentials and see if we obtain the same solution.)
(a) We set

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} := \begin{pmatrix} y(t) \\ y'(t) \\ y''(t) \end{pmatrix}$$

Then $y'''(t) + 4y''(t) - 4y'(t) = 0$ becomes

$$(*) \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & -4 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix}$$

Let us call the matrix in this equation A.

(b) Diagonalizing A as usual gives:

$$A = SDS^{-1} = \begin{pmatrix} 1 & 3+2\sqrt{2} & 3-2\sqrt{2} \\ 0 & 2+2\sqrt{2} & -2\sqrt{2}+2 \\ 0 & 4 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\sqrt{2}-2 & 0 \\ 0 & 0 & -2-2\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & -1 & -\frac{1}{4} \\ 0 & \frac{1}{4\sqrt{2}} & -\frac{1-\sqrt{2}}{8\sqrt{2}} \\ 0 & -\frac{1}{4\sqrt{2}} & \frac{2+\sqrt{2}}{16} \end{pmatrix}$$

Hence

$$\begin{aligned}
\exp(tA) &= S\exp(tD)S^{-1} \\
&= \begin{pmatrix} 1 & 3+2\sqrt{2} & 3-2\sqrt{2} \\ 0 & 2+2\sqrt{2} & -2\sqrt{2}+2 \\ 0 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{(2\sqrt{2}-1)t} & 0 \\ 0 & 0 & e^{-2(1+\sqrt{2})t} \end{pmatrix} \begin{pmatrix} 1 & -1 & -\frac{1}{4} \\ 0 & \frac{1}{4\sqrt{2}} & -\frac{1-\sqrt{2}}{8\sqrt{2}} \\ 0 & -\frac{1}{4\sqrt{2}} & \frac{2+\sqrt{2}}{16} \end{pmatrix}
\end{aligned}$$

(Technically, since the problem says "Compute $\exp(tA)$", you we are asked to simplify, i.e. compute the threefold matrix product. But as it is quite lengthy and not very enlightening, let us skip writing it out here.)

(c) The solution of $(*)$ for the initial value

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = S\exp(tD)S^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3+2\sqrt{2} & 3-2\sqrt{2} \\ 0 & 2+2\sqrt{2} & -2\sqrt{2}+2 \\ 0 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{(2\sqrt{2}-1)t} & 0 \\ 0 & 0 & e^{-2(1+\sqrt{2})t} \end{pmatrix} \begin{pmatrix} 1 & -1 & -\frac{1}{4} \\ 0 & \frac{1}{4\sqrt{2}} & -\frac{1-\sqrt{2}}{8\sqrt{2}} \\ 0 & -\frac{1}{4\sqrt{2}} & \frac{2+\sqrt{2}}{16} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(If you compute the above matrix product starting from right to left, then it is actually a short computation!!)

Hence $y(t) = x_1(t) = 1$ is the solution of $y'''(t) - 4y''(t) + 4y'(t) = 0$ for the initial values $y''(0) = y'(0) = 0, y(0) = 1$.

Ex 14.10 ($\exp(tA)$ for a non-diagonalizable matrix)

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. a) Show that A is not diagonalizable.

b) Show by induction that $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

c)* *Non-mandatory exercise.* Show that $\exp(tA) = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$.

You may use the formula $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Solution:

a) A being an upper triangular matrix, we see that its only eigenvalue is 1, but the corresponding eigenspace cannot be two-dimensional since otherwise the matrix $A - I_2$ has to be zero matrix. Hence A is not diagonalizable.

b) For $n = 0$ the statement holds since $A^0 := I_2$. Now suppose it holds for $n-1$ for $n \geq 1$. Then

$$A^n = AA^{n-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (n-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

c) By part b) and the definition of the exponential series we have

$$\exp(tA) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}}{n!} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{t^n}{n!} & \sum_{n=0}^{\infty} \frac{t^n n}{n!} \\ 0 & \sum_{n=0}^{\infty} \frac{t^n}{n!} \end{pmatrix} = \begin{pmatrix} e^t & \sum_{n=0}^{\infty} \frac{t^n n}{n!} \\ 0 & e^t \end{pmatrix}.$$

There we only have to identify the entry at position $(1,2)$. It holds that

$$\sum_{n=0}^{\infty} \frac{t^n n}{n!} = \sum_{n=1}^{\infty} \frac{t t^{n-1}}{(n-1)!} = t \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} \overset{k=n-1}{=} t \sum_{k=0}^{\infty} \frac{t^k}{k!} = te^t.$$

Remark: This shows that the solution of the ODE system $x_1' = x_1 + x_2$ and $x_2' = x_2$ with initial values $x_1(0) = 0$ and $x_2(0) = 1$ is given by $x_1(t) = te^t$ and $x_2(t) = e^t$, so that it is no pure linear combination of exponential functions.

Ex 14.11 (Diagonalization of a matrix exponential)
Let A be the matrix from Exercise 11.2 (see Homework 11). Diagonalize $\exp(tA)$ for $t \in \mathbb{R}$.

Solution: From the solution of Exercise 11.2: $A = PDP^{-1}$ where

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, P = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

So we can compute

$$P^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

Moreover, by Theorem 7.9: $\exp(tA) = P\exp(tD)P^{-1}$ So

$$\exp(tA) = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

Since we were asked to *diagonalize* $\exp(tA)$ (as opposed to *computing* it) we do not have to further simplify.

Ex 14.12 (Multiple choice and True/False questions)

a) Consider the matrices $A = \begin{pmatrix} -7/4 & 1/2 \\ 1/2 & 3/5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

   Which among the following statements are true?

   (A) A and B are orthogonally diagonalizable
   (B) A is orthogonally diagonalizable and B is diagonalizable
   (C) A is diagonalizable but B is not.
   (D) neither A nor B are orthogonally diagonalizable.

b) Decide whether the following statements are always true or if they can be false.

   (i) If $A = A^T$ and $Ax = 0$ and $Ay = y$, then $x \cdot y = 0$.
   (ii) If $A = A^T$, then A has n distinct real eigenvalues.
   (iii) An orthogonal matrix is orthogonally diagonalizable.
   (iv) If $A \in \mathbb{R}^{m \times n}$ and if $P \in \mathbb{R}^{m \times m}$ is orthogonal, then A and PA have the same singular values.
   (v) If $A \in \mathbb{R}^{n \times n}$, then A and $A^T A$ have the same singular values.

(vi) If A is orthogonally diagonalizable, then $\exp(tA)$ is orthogonally diagonalizable.

Solution:
a) **(B)**: The matrix A is symmetric, so it is orthogonally diagonalizable, while a quick calculation shows that B has two distinct eigenvalues, so it is diagonalizable.
As a consequence, **(C)** and **(D)** are false. **(A)** is also false because B is not symmetric, hence it cannot be orthogonally diagonalizable.

b) True/false

(i) **TRUE**: If either $x = 0$ or $y = 0$, then the orthogonality is clear. Otherwise they are eigenvectors for the two eigenvalues 0 and 1, so they are orthogonal since A is symmetric.

(ii) **FALSE**: Take $A = I_n$, which is symmetric and has only the eigenvalue 1.

(iii) **FALSE**: It suffices to find an orthogonal matrix that is not symmetric. Take for instance $Q = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

(iv) **TRUE**: Recall that the singular values of A are the eigenvalues of A^TA. When P is orthogonal, we have $(PA)^T(PA) = A^TP^TPA = A^TA$.

(v) **FALSE**: Consider $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Then $A^TA = A^2$ has the eigenvalue 4, so that A has the singular value 2, while $(A^TA)^TA^TA = A^4$ has the eigenvalue 16, so that A^TA has the singular value 4.

(vi) **TRUE** If A is orthogonally diagonalizable, this means that $A = PDP^{-1}$ where D is diagonal and P is orthogonal. By Thm.7.9, it follows that $\exp(tA) = P\exp(tD)P^{-1}$ and by Lemma 7.8 $\exp(tD)$ is diagonal. Hence this is an orthogonal diagonalization of $\exp(tA)$.